Selected risk-sensitive optimal stopping and impulse control problems

PhD Thesis

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Chapter 1

Introduction

In this thesis, we consider several classes of stochastic control problems with the risk-sensitive optimality criterion. We focus on optimal stopping and impulse control problems, and, using purely probabilistic methods, we characterise optimal strategies for a wide class of Feller–Markov processes. Following the risk-sensitive approach, we contribute to the theory of non-linear stochastic control problems, which facilitate better stability of results compared to the classic risk-neutral (linear) framework.

Stochastic control theory looks for an optimal way of controlling some underlying phenomenon that is subject to random perturbations. A generic form of a stochastic control problem could be expressed as

$$\sup_{a\in\mathcal{A}} J(Z(a)), \tag{1.0.1}$$

where a control a from a set of admissible controls $\mathcal{A}$ affects a distribution of a random outcome $Z(a)$ and should maximise some optimality criterion J; see e.g. Bertsekas and Shreve (1978); Pham (2009), and Bäuerle and Rieder (2011) for a comprehensive overview. In the financial context, the control $a \in \mathcal{A}$ could describe an admissible investment strategy, and $Z(a)$ could correspond to the associated cash-flow process. Typically, the functional J is used to quantify the performance (utility) of the underlying project. It should be highlighted that, in most cases, the problem is dynamic, i.e. the performance of the project is affected by some stochastic process, and the strategy should take into account this time dependence. In particular, an optimal strategy should dynamically adjust to the changing environment.

In the classic context, the functional J corresponds to the expectation operator (or expected utility) which leads to linear problems; see e.g. Bäuerle and Rieder (2011). In this thesis, the optimality criterion J is associated with the so-called *entropic (risk-sensitive) utility measure* defined as

$$J^{\gamma}(Z) := \begin{cases} \frac{1}{\gamma}\ln \mathbb{E}\left[\exp(\gamma Z)\right], & \gamma \neq 0, \\ \mathbb{E}\left[Z\right], & \gamma = 0, \end{cases} \tag{1.0.2}$$

where Z is a random variable corresponding to the pay-off (or logarithmic growth) of some investment project. The parameter $\gamma \in \mathbb{R}$ in (1.0.2) reflects an investor's risk-aversion. The case $\gamma < 0$ (respectively $\gamma > 0$) describes the preferences of the *risk-averse* (respectively *risk-seeking*) investor, while $\gamma = 0$ corresponds to the *risk-neutral* preferences; see Howard and Matheson (1972) for a seminal discussion. In this thesis, we focus on J^{γ} with $\gamma < 0$, which is referred to as the *risk-sensitive criterion*.

For $\gamma < 0$, the functional (1.0.2) may be seen as the certainty equivalent for the exponential utility function. Also, this functional could be seen as a non-linear extension of the classic linear-quadratic Markowitz criterion. In the Markowitz approach, the risk of the associated project is measured only by the variance. However, it can be shown that the risk-sensitive criterion also takes into account higher-order moments, which facilitate better stability of the results; see e.g. Davis and Lleo (2014) for details. In fact, this criterion could be seen as a non-linear extension of some other classic optimality criteria studied in the stochastic control literature; see e.g. Hernandez-Lerma and Lasserre (1996). In particular, for Z corresponding to the logarithmic rate of return, the functional (1.0.2) could be seen as a non-linear version of the celebrated Kelly criterion, which is frequently used by practitioners; see MacLean et al. (2011).

The interesting properties of the risk-sensitive criterion result in the extensive literature on this topic. In particular, this refers to generic stochastic control problems and Markov decision processes, see e.g. Hernández-Hernández and Marcus (1996); Di Masi and Stettner (1999); Borkar and Meyn (2002); Cavazos-Cadena and Montes-De-Oca (2005); Di Masi and Stettner (2007); Jaśkiewicz (2007); Bäuerle and Rieder (2014); Cavazos-Cadena and Hernández-Hernández (2017) for the discussion in the discrete time setting, and e.g. Fleming and McEneaney (1995); Menaldi and Robin (2005); Biswas (2011); Arapostathis and Biswas (2018, 2020) for the analysis in the continuous time framework. Also, risk-sensitive problems are considered in the partial observation framework, see e.g. Hernández-Hernández (1999); Cavazos-Cadena and Hernández-Hernández (2005); Bäuerle and Rieder (2017). From the applied point of view, risk-sensitive problems are studied in connection with portfolio management, see e.g. Bielecki and Pliska (1999); Fleming and Sheu (2000); Nagai (2003); Pitera and Stettner (2016), as well as various applications in other disciplines, e.g. missile guidance, reinforcement learning, or neuroscience; see e.g. Speyer (1976); Mihatsch and Neuneier (2002); Nagengast et al. (2010). Nevertheless, it should be noted that, in the context of the risk-sensitive setting, the applications of standard methods from the risk-neutral case often lead to complex problems, typically involving quasi variational inequalities; see e.g. Nagai (2006); Davis et al. (2010); Belak et al. (2017); Arapostathis and Biswas (2020).

In this thesis, we focus on two types of controls, that is, optimal stopping and impulse interventions. This choice corresponds to the fact that these controls are usually the only feasible solutions to real-world problems. Moreover, there is an intrinsic mathematical connection between impulse control and optimal stopping; see e.g. Robin (1981).

In the following two paragraphs, we provide an overview of the problems considered in this thesis. In particular, we briefly introduce the main objectives and discuss the related literature. Then, we comment on the generic structure of the thesis and outline the key results.

Optimal stopping problems The first type of controls considered in this thesis is associated with optimal stopping times. In this context, the family of admissible controls $\mathcal{A}$ from (1.0.1) consists of stopping times τ, and the random variable $Z(\tau)$ quantifies the performance of the project terminated (stopped) at τ. It should be noted that many practical control problems could be expressed in terms of optimal stopping. This includes some important examples in mathematical finance (e.g. American options theory, optimal asset liquidation), statistics (sequential testing), operations research, and ecology; see e.g. Shiryaev (1978); Bensoussan and Lions (1984); Carmona and Touzi (2008); Bäuerle and Rieder (2011) for details.

With reference to the risk-sensitive optimal stopping framework, we study problems of the form

$$\sup_{\tau} J_x^{\gamma}\left(\int_0^{\tau} \widehat{g}(X_s)ds + \widehat{G}(X_\tau)\right), \quad x \in E, \tag{1.0.3}$$

where τ is a stopping time, X is a standard Markov process with values in a state space E, J_x^{γ} is a version of (1.0.2), where the expectation operator $\mathbb{E}$ is replaced by the conditional expectation given that $X_0 = x$, while $\widehat{g}$ and $\widehat{G}$ are running cost and terminal reward/cost functions, respectively. From the economic perspective, (1.0.3) encodes the problem of a decision-maker who wants to find an optimal time for a termination of some investment project. As long as the project is active, the decision-maker has to cover the running cost (associated with $\widehat{g}$), which may reflect some operational or financial expenses. At the moment of stopping, the decision-maker gets the terminal reward (measured by $\widehat{G}$), which may correspond to the contingent claim pay-off or value of the sold asset.

Regarding (1.0.3), we study the structure of an optimal stopping time and the regularity properties of the value function. In particular, we investigate the continuity with respect to a starting point x and various approximation schemes, including time horizon limits and dyadic approximations. Our analysis is based on the study of the corresponding optimal stopping Bellman equation. In particular, we identify the extremal solutions to this equation and give a sufficient condition for the uniqueness.

Typically, a characterisation of the optimal stopping times is based on the corresponding Snell envelope of the value process; see e.g. Bismut and Skalli (1977) and El Karoui (1981) for classic contributions, and Kobylanski and Quenez (2012) for more recent results. Also, to obtain an optimal stopping time, it is possible to use convex optimisation arguments and duality theory; see e.g. Pennanen and Perkkiö (2019). In the Markovian setting, many optimal stopping problems could be solved with the help of a specific optimality Wald–Bellman equation; see e.g. Shiryaev (1978); Peskir and Shiryaev (2006); Shiryaev (2019). The existence of a solution to this equation could be obtained by value iteration arguments or the penalty approach, see e.g. Zabczyk (1984) and Stettner (2011). Also, it may result from the use of viscosity techniques applied to variational inequalities; see e.g. Bensoussan and Lions (1984); Pham (2009), and Dai and Menoukeu-Pamen (2018).

In the literature, regularity properties of the optimal stopping value function were primarily studied in the context of risk-neutral (additive) stopping problems; see e.g. Bassan and Ceci (2002) and Palczewski and Stettner (2014). In particular, this applies to the non-uniqueness of a solution to the Bellman equation; see Section 2.11 in Shiryaev (1978) and Theorem 1.13 in Peskir and Shiryaev (2006) for classic contributions. In the risk-sensitive context, Nagai (2007*b*) considered a variational characterisation of the optimal stopping value function, Bäuerle and Rieder (2015) studied the properties of partially observable discrete time stopping problem, and Bäuerle and Popp (2018) considered stopping problem for piecewise constant Markov process (continuous time Markov chain). It should be noted that, typically, the risk-sensitive stopping problems cannot be easily embedded in the classic optimal stopping framework. This could be associated with the more complex, multiplicative nature of these problems; see e.g. Nagai (2007*b*) and Section 3.1 in this thesis for further details.

Impulse control problems The second type of controls studied in this thesis is associated with impulse interventions. In a nutshell, an impulse control strategy $V := (\tau_i, \xi_i)_{i=1}^{\infty}$ for the stochastic process X is described by the increasing sequence of impulse times (τ_i) and the sequence of after-impulse states (ξ_i). Whenever a decision-maker decides to apply an impulse, the process is shifted to the chosen state. More specifically, up to the time τ_1, the process X evolves according to its usual dynamics, at τ_1 it is shifted to ξ_1, where it starts its evolution again and follows it up to the time of the next impulse τ_2, etc. We refer to Robin (1978) and Section 2.2 in this thesis for a more detailed discussion.

It should be noted that impulse controls are widely used to solve some practical problems. In this way, one can affect a continuous time phenomenon in the discrete time manner, taking into account the delay between the decision and its execution, infrastructure capacity (e.g. limit order book characteristics at the stock exchange), and transaction costs; see e.g. Bensoussan and Lions (1984) and Davis (1993) for a detailed discussion. From a practical point of view, this type of control may be applied e.g. to design an optimal intervention scheme for a central bank on the foreign exchange market. In this context, the controlled process describes the exchange rate, and the decision-maker needs to determine when to intervene (impulse times) and what is the target rate (after-impulse states); we refer to Jeanblanc-Picqué (1993); Runggaldier and Yasuda (2018), and references therein for a detailed discussion. Other applications of impulse control strategies include i.a. controlling epidemics, ecosystems (optimal harvesting schemes), finance (cash management), and portfolios with transaction costs; see e.g. Piunovskiy et al. (2020); Erdlenbruch et al. (2013); Córdova-Lepe et al. (2012); Korn (1999).

The practical importance of the impulse control results in the extensive literature on this subject, see e.g. Robin (1981); Davis et al. (2010); Palczewski and Stettner (2010); Bayraktar et al. (2013); Dufour and Piunovskiy (2016); Menaldi and Robin (2017); Palczewski and Stettner (2017). Still, its coverage in the risk-sensitive case is limited and only very specific cases have been studied so far. In particular, Sadowy and Stettner (2002) considered continuous time Markov processes with additive shift-cost functions, Nagai (2007*a*) discussed the problem of optimal investment with transaction costs, Hdhiri and Karouf (2011) considered general non-Markovian setting with impulse cost depending only on the after-impulse state, and Pitera and Stettner (2021) proved the existence of a solution to the impulse control optimality equation in the dyadic framework.

With reference to the impulse control framework, we study problems of the form

$$\sup_{V} \liminf_{T\to\infty} \frac{1}{T} J^{\gamma}_{(x,V)} \left(\int_0^T \widehat{f}(Y_s)ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} \widehat{c}(Y_{\tau_i^-}, \xi_i) \right), \quad x \in E, \quad (1.0.4)$$

where $J^{\gamma}_{(x,V)}$ is a conditional version of the entropic utility measure (1.0.2) corresponding to a starting point x and a strategy V, the function $\widehat{f}$ describes the running reward, while $\widehat{c}$ measures the cost associated with impulses, and $Y_{\tau_i^-}$ describes the state of the controlled process right before the impulse.

Problems of the type of (1.0.4) could be seen as a maximisation of the long run logarithmic rate of return corresponding to the impulse control policy; see Bielecki and Pliska (2003). Typically, they are very robust and time-consistent. In particular, the solution is often independent of the starting point x (ergodic property); see Hernandez-Lerma and Lasserre (1996) for a detailed discussion.

It should be noted that impulse control and optimal stopping problems are tightly linked. Typically, the suitable optimal stopping times identify the right moments when one should apply the impulse; see Robin (1978). Thus, while studying impulse control problems, we extensively use the obtained results on the optimal stopping.

Main results This thesis is based on three published papers, i.e. Jelito et al. (2020, 2021) and Jelito and Stettner (2022). The main contributions of the thesis are as follows. First, we study the risk-sensitive optimal stopping Bellman equations in the discrete and continuous time settings. In particular, we provide a probabilistic interpretation for the minimal and the maximal solutions to these equations in terms of the suitable infinite time horizon stopping problems; see Theorem 3.2.6 and Theorem 3.4.3. The results cover both discrete and continuous time frameworks. It should be noted that the continuous time case requires the use of some non-standard techniques related to the properties of Feller-Markov processes.

Second, under relatively weak assumptions imposed on the underlying process, we obtain several results related to the regularity properties of the infinite time horizon optimal stopping value functions. In particular, in Theorem 3.2.14 and Theorem 3.4.11, we show that the value functions are continuous with respect to a starting point of the process and we provide formulae for optimal stopping times. The results are linked to the uniqueness of a solution to the optimal stopping Bellman equation. Thus, in Theorem 3.2.11 and Theorem 3.4.7, we provide generic sufficient conditions for this property.

Third, while studying infinite time horizon optimal stopping problems, we obtain numerous auxiliary results that are interesting on their own merit. In particular, in Section 3.3, we show that the suitable finite time horizon stopping problems are jointly continuous with respect to a starting point and a time horizon; see Theorem 3.3.2. In the proof, we introduce a dyadic approximation of the value function that could be used to obtain a numerical approximation scheme for the problem. Also, the proof is based on the distance control properties of the wide class of Feller-Markov processes which, to our best knowledge, have not been considered in the risk-sensitive context.

Fourth, we study finite and infinite time horizon (long-run) risk-sensitive impulse control problems. In both cases, we formulate suitable forms of the associated impulse control Bellman equations and prove the existence of solutions to them; see Theorem 4.2.5 and Theorem 4.3.8. Based on these results, we construct optimal strategies for the corresponding problems; see Theorem 4.2.2 and Theorem 4.3.4. The argument for the finite time horizon case is linked to the problems with finitely many impulses. In this way, we obtain several approximation results that can be used to numerically solve the underlying problem; see Proposition 4.2.9. The argument for the long-run case uses a time discretisation technique. More specifically, we consider impulse control problems on the dyadic time-grid and show that they approximate the initial problem when the time step goes to zero. To do this, we link the underlying impulse control problems to the suitable optimal stopping setting and use the obtained results on the risk-sensitive optimal stopping. The specific link is based on the change of measure transformation associated with a solution to the Multiplicative Poisson Equation; see Proposition 4.3.2 and Theorem 4.3.8.

Fifth, this thesis contains a detailed analysis of the computable examples illustrating our assumptions and results. In particular, we show that our framework covers a wide class of Feller-Markov processes, including piecewise deterministic dynamics and reflected diffusions; see Section 5.1 and Section 5.2. Also, in Section 5.3, we present a toy dynamics with explicit multiple solutions to the optimal stopping Bellman equations, both in the discrete and continuous time settings; see Example 5.3.1 and Example 5.3.5. This shows that some of our results cannot be generalised.

Sixth, for the reader's convenience, we provide a comprehensive comment on selected techniques used in this thesis. In particular, in Section A.4, we give a detailed description of the change of measure transformation based on the solution to the Multiplicative Poisson Equation. The idea that we may simplify some optimisation problems with the help of the change of measure was used e.g. in Robin (1981) in the additive setting and in Sadowy and Stettner (2002) in the multiplicative case. However, usually, the properties of the new measures are only outlined. In this thesis, we clarified some points related to the corresponding Feller-Markov process; see Theorem A.4.6 for details. This can be used to further generalise the use of the change of measure technique.

Finally, this thesis contains several auxiliary methodological ideas that, in our opinion, are worth mentioning. First, we would like to point out Lemma 3.3.1, which, in our belief, contains interesting applications of the distance control properties of Feller-Markov processes. This simplifies estimates related to the uniform convergence of specific expectations and is one of the main building blocks of the proof of Theorem 3.3.2. A similar technique is used in the proof of Proposition 4.2.8. Also, it is worth highlighting the proof of the first part of Theorem 3.4.3, which contains a non-standard argument related to the convergence of stopping times. More specifically, using the structure of the corresponding optimal stopping value functions, we can find an optimal stopping time for the minimal solution to the Bellman equation and link it to the suitable finite time horizon problem. A similar technique is used in the proof of Theorem 3.5.1. Lastly, we would like to highlight again the change of measure transformation based on the Multiplicative Poisson Equation. In particular, in Proposition 4.3.2, we use this technique to transform the associated stopping problem into the setting considered in Chapter 3. We believe that this is an interesting approach that can be used to analyse some other stopping problems that do not satisfy standard assumptions.

Structure The structure of this thesis is as follows. This introduction is followed by Chapter 2, where we set up the terminology and notation used throughout this thesis. Next, in Chapter 3, we discuss the optimal stopping problems and, in Chapter 4, we consider the impulse control problems. Also, in Chapter 5, we present a series of examples for our assumptions and results. In Appendix A, we discuss some complementary results. The thesis concludes with the list of references and the index of frequently used symbols.

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Chapter 2

Preliminaries

In this chapter, we introduce some basic terminology and notation that is used throughout this thesis. In particular, in Section 2.1, we discuss the notion of a standard Markov process and comment on the regularity properties of its transition semigroup. Next, in Section 2.2, we review the construction of the process corresponding to the impulse control strategy. Finally, in Section 2.3, we discuss the properties of the entropic utility measure and the risk-sensitive criterion. This chapter is intended to introduce the terminology and recall some useful results; no claim of originality is made here.

Before we proceed, we summarise some conventions and notation used in this thesis. Given some set A, by A^c we denote its complement. By $\mathbb{N}$ and $\mathbb{R}_+$ we denote the sets of non-negative integers and non-negative real numbers, respectively. Also, we define $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$. For $x, y \in \mathbb{R}$, by $x \wedge y$ we denote their minimum, i.e. $\min(x, y)$, with a natural (pointwise) extension for functions and random variables. If not stated otherwise, all equalities associated with random variables should be understood in the almost sure sense with respect to the underlying probability measure. That saying, we sometimes write "μ a.s." to emphasize that some property holds almost surely with respect to a specific measure μ. We follow the convention that the infimum of the empty set is equal to $+\infty$ and we set the value of an "empty" sum to 0, i.e. we have $\sum_{i=0}^{-1}(\cdot) := 0$. The end of an example and a remark is denoted by ⧫ and $\lozenge$, respectively.

In the thesis we extensively use some notation related to families of sets and functions. Given a metric space A, by $\mathcal{B}(A)$ we denote the Borel σ-field on A. Next, by $\mathcal{M}(A)$ and $\mathcal{C}(A)$ we denote the family of $\mathcal{B}(A)$-measurable real-valued functions on A and continuous real-valued functions on A, respectively. Also, by $\mathcal{M}_b(A) \subset \mathcal{M}(A)$ and $\mathcal{M}^+(A) \subset \mathcal{M}(A)$ we denote the corresponding subfamilies of bounded and non-negative functions, respectively. Similarly, by $\mathcal{C}_b(A) := \mathcal{C}(A) \cap \mathcal{M}_b(A)$ and $\mathcal{C}^+(A) := \mathcal{C}(A) \cap \mathcal{M}^+(A)$ we denote the corresponding subfamilies of bounded continuous and non-negative continuous functions, respectively. Also, we set $\mathcal{C}_b^+(A) := \mathcal{C}_b(A) \cap \mathcal{C}^+(A)$. Next, by $\mathcal{C}_0(A)$ we denote the family of continuous functions vanishing at infinity, i.e. we get that $f \in \mathcal{C}_0(A)$ if, for any $\varepsilon > 0$, we may find a compact set $K \subseteq A$ such that $|f(x)| \leq \varepsilon$ for $x \notin K$. Also, for any map $f \in \mathcal{M}_b(A)$, we define its supremum norm by $\|f\| := \sup_{x \in A} |f(x)|$. Next, note that in this thesis, monotonicity properties are understood in the weak sense, i.e. a map $f\colon \mathbb{R} \to \mathbb{R}$ is called increasing (respectively, decreasing) if, for any $x < y$, we get $f(x) \leq f(y)$ (respectively, $f(x) \geq f(y)$). Finally, for a real sequence $(a_n)_{n=1}^{\infty}$ and $a \in \mathbb{R} \cup \{-\infty, +\infty\}$, by $a_n \uparrow a$ (respectively, $a_n \downarrow a$) we denote the fact that (a_n) converges increasingly (respectively, decreasingly) to a as $n \to \infty$.

2.1 Feller-Markov processes

In this section, we recall the notion of a standard Markov process and its basic properties. Also, we discuss specific regularity properties of the transition semigroup associated with this process. This section is based mainly on Section 1.4 of Shiryaev (1978); see also Dynkin (1965); Gikhman and Skorokhod (1975), and Blumenthal and Getoor (2007) for further discussion.

Let us start with establishing the necessary notation. By $\mathbb{T}$ we denote the set of considered time points; in this thesis this is either $\mathbb{T} = \mathbb{N}$ in the discrete time setting or $\mathbb{T} = \mathbb{R}_+$ in the continuous time case. Also, by $(\Omega, \mathcal{F}, \mathbb{F})$ we denote a filtered measurable space with a set of elementary outcomes Ω, a σ-field $\mathcal{F}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$. We assume that the filtration is right-continuous and universally complete. Next, let $(E, \mathcal{E})$ be a locally compact separable metric space with a metric ρ and the Borel σ-field $\mathcal{E} := \mathcal{B}(E)$. The pair $(E, \mathcal{E})$ constitutes a state space for a Markov process which is defined below. Typically, E is a subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}_*$ with the Euclidean metric. From now on we fix some $x_0 \in E$ and, with a slight abuse of notation, we define a norm-like function $\|x\| := \rho(x_0, x)$, $x \in E$. In particular, if E is a vector space, we choose $x_0 = 0$ and $\|\cdot\|$ corresponds to the norm induced by the metric ρ.

2.1.1 Standard Markov processes

The main focus of this thesis is set on the control problems related to Markov processes. In fact, we focus on the so-called *standard Markov processes*. Before we provide a formal definition, let us outline the main idea. A process (X_t), $t \in \mathbb{T}$, on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$ is Markov (with respect to a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$), if, for any $B \in \mathcal{E}$ and $t, s \in \mathbb{T}$, we have

$$\mathbb{P}\left[X_{t+s} \in B | \mathcal{F}_t\right] = \mathbb{P}\left[X_{t+s} \in B | X_t\right] \quad \mathbb{P}\,\text{a.s.} \tag{2.1.1}$$

Intuitively, Equation (2.1.1) states that the distribution of X_{t+s} conditioned on the history encoded by $\mathcal{F}_t$ depends only on the present state of the process X_t. Based on this property we may consider a process restarted at X_t and heuristically rewrite (2.1.1) as

$$\mathbb{P}\left[X_{t+s} \in B | \mathcal{F}_t\right] = \mathbb{P}\left[X_s \in B | X_0 = X_t\right] \quad \mathbb{P}\,\text{a.s.} \tag{2.1.2}$$

To make this formula formally meaningful, in Definition 2.1.3 we introduce the notion of a standard Markov process. The main difference between this concept and (2.1.1) lies in the fact that now, instead of a single measure $\mathbb{P}$, we consider a family of probability measures $\mathbb{P}_x$, $x \in E$, which are linked to X. In this way, we may provide a rigorous meaning of (2.1.2). Also, standard Markov processes are characterised by several useful properties that facilitate the analysis of control problems; see the discussion following the definition for details.

Definition 2.1.1. A pair $((X_t)_{t\in\mathbb{T}}, (\mathbb{P}_x)_{x\in E})$ is called a *standard Markov process* on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in the *state space* $(E, \mathcal{E})$ if the following hold:

(1) The process (X_t), $t \in \mathbb{T}$, takes values in $(E, \mathcal{E})$, has càdlàg trajectories and is adapted to $\mathbb{F}$.

(2) For any $x \in E$, the map $\mathcal{F} \ni A \mapsto \mathbb{P}_x(A)$ is a probability measure and we have $\mathbb{P}_x[X_0 = x] = 1$. Also, for any $A \in \mathcal{F}$, the map $E \ni x \mapsto \mathbb{P}_x(A)$ is measurable.

(3) The process (X_t), $t \in \mathbb{T}$, satisfies the *strong Markov property*, i.e. for any $x \in E$, $B \in \mathcal{E}$, $s \in \mathbb{T}$, and a stopping time τ satisfying $\mathbb{P}_x[\tau < \infty] = 1$, we have
$$\mathbb{P}_x\left[X_{\tau+s} \in B | \mathcal{F}_\tau\right] = \mathbb{P}_{X_\tau}\left[X_s \in B\right] \quad \mathbb{P}_x\,\text{a.s.} \tag{2.1.3}$$

(4) The process (X_t), $t \in \mathbb{T}$, is *quasi left-continuous*, i.e. for any $x \in E$, a stopping time τ satisfying $\mathbb{P}_x[\tau < \infty] = 1$ and any increasing sequence of stopping times (τ_n) such that $\tau_n \uparrow \tau$ as $n \to \infty$, we have $X_{\tau_n} \to X_\tau$ $\mathbb{P}_x$ a.s.

(5) For any $t \in \mathbb{T}$ and $\omega \in \Omega$, there exists $\widetilde{\omega} \in \Omega$ such that, for any $s \in \mathbb{T}$, we get $X_{t+s}(\omega) = X_s(\widetilde{\omega})$.

Condition (1) is a standard assumption for the adaptedness and regularity of the process trajectories. Based on Condition (2) we may interpret $\mathbb{P}_x$ as a conditional distribution given that $X_0 = x$, $x \in E$. Condition (3) states the strong Markov property of (X_t), $t \in \mathbb{T}$, and gives a precise meaning of (2.1.2). Note that if (3) holds only for deterministic stopping times, the resulting process is called (weak) Markov; cf. (2.1.1). Condition (4) requires the quasi left-continuity of the process; it should be noted that the trajectories may be discontinuous since we assumed only the càdlàg property. Also, note that Conditions (3) and (4) are automatically satisfied in the discrete time setting, i.e. for $\mathbb{T} = \mathbb{N}$; see Section 3.18 in Dynkin (1965) for details. Condition (5) implies that we have a sufficiently "rich" set Ω and can be used to define the Markov shift operator; see Section 1.4.3 in Shiryaev (1978) for details. Typically, this condition reflects the properties of a canonical probability space. In fact, when necessary, we assume that the underlying probability space is a canonical space of càdlàg processes with a suitable (Borel) σ-field.

For brevity, if $\mathbb{T} = \mathbb{N}$, we say that $X = (X_n)_{n\in\mathbb{N}}$ is a discrete time standard Markov process and if $\mathbb{T} = \mathbb{R}_+$, we say that $X = (X_t)_{t\in\mathbb{R}_+}$ is a continuous time standard Markov process. In any case, we tacitly assume that the family of probability measures $(\mathbb{P}_x)_{x\in E}$ is given and, together with X, satisfies Definition 2.1.1. Also, for any $x \in E$, by $\mathbb{E}_x$ we denote the expectation operator corresponding to $\mathbb{P}_x$.

One may consider a more general setting than the one stated in Definition 2.1.1. In particular, we may define a time-inhomogeneous Markov process which conditional distribution depends on the current time and the current state. It can be shown that this process may be embedded in the time-homogeneous setting by a suitable enlargement of the state space; see e.g. Section 1.4.6 in Shiryaev (1978) for details.

Most of the results in this thesis are associated with optimal stopping problems. Thus, let us now introduce some notation related to stopping times. To simplify the narrative, we fix some standard Markov process $((X_t)_{t\in\mathbb{T}}, (\mathbb{P}_x)_{x\in E})$ on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. In the continuous time setting (i.e. for $\mathbb{T} = \mathbb{R}_+$), by $\mathcal{T}$ we denote the family of stopping times with respect to the filtration $\mathbb{F}$ taking values in $\mathbb{R}_+ \cup \{+\infty\}$. Next, for any $x \in E$, by $\mathcal{T}_x$ and $\mathcal{T}_{x,b}$ we denote the families of $\mathbb{P}_x$ a.s. finite and $\mathbb{P}_x$ a.s. bounded stopping times, respectively. Also, we consider stopping times with values on the dyadic time-grid. Thus, setting $\delta_m := \frac{1}{2^m}$, $m \in \mathbb{N}$, for any $m \in \mathbb{N}$ we define $\mathcal{T}^m$ as the family of stopping times with values in the set $\{k\delta_m \colon k \in \mathbb{N}\} \cup \{+\infty\}$. Also, for any $m \in \mathbb{N}$ and $x \in E$, we define $\mathcal{T}_x^m := \mathcal{T}^m \cap \mathcal{T}_x$ and $\mathcal{T}_{x,b}^m := \mathcal{T}^m \cap \mathcal{T}_{x,b}$. In particular, $\mathcal{T}_x^0$ and $\mathcal{T}_{x,b}^0$ denotes the respective family of stopping times with values in $\mathbb{N}$. Also, with a slight abuse of notation, we use $\mathcal{T}_x^0$ and $\mathcal{T}_{x,b}^0$ to denote the respective families of $\mathbb{P}_x$ a.s. finite and $\mathbb{P}_x$ a.s. bounded stopping times in the discrete time framework (i.e. for $\mathbb{T} = \mathbb{N}$).

2.1.2 Transition semigroup and the Feller property

In this section, we introduce the transition semigroup linked to the standard Markov process and discuss its basic properties. Throughout this section, by $((X_t)_{t\in\mathbb{T}}, (\mathbb{P}_x)_{x\in E})$, we denote a standard Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Also, we recall that we have $\mathbb{T} = \mathbb{N}$ in the discrete time setting and $\mathbb{T} = \mathbb{R}_+$ in the continuous time case.

For any $t \in \mathbb{T}$, we define the transition operator $\mathcal{P}_t \colon \mathcal{M}_b(E) \to \mathcal{M}_b(E)$ associated with X by

$$\mathcal{P}_t h(x) := \mathbb{E}_x[h(X_t)], \quad h \in \mathcal{M}_b(E),\, x \in E. \tag{2.1.4}$$

Based on Condition (3) in Definition 2.1.1, we get that $\mathcal{P}_t \circ \mathcal{P}_s = \mathcal{P}_{t+s}$, $t, s \in \mathbb{T}$, thus $(\mathcal{P}_t)_{t\in\mathbb{T}}$ forms a semigroup. In fact, using Condition (2) from Definition 2.1.1, we also get $\mathcal{P}_0 h \equiv h$, thus $(\mathcal{P}_t)_{t\in\mathbb{T}}$ is a monoid with the identity element $\mathcal{P}_0$.

Now, we define several useful classes of the regularity of the semigroup $(\mathcal{P}_t)_{t\in\mathbb{T}}$.

Definition 2.1.2. We say that the transition semigroup $(\mathcal{P}_t)_{t\in\mathbb{T}}$ is

(1) $\mathcal{C}_b$-*Feller* if $P_t\mathcal{C}_b(E) \subset \mathcal{C}_b(E)$ for any $t \in \mathbb{T} \setminus \{0\}$;

(2) $\mathcal{C}_0$-*Feller* if $P_t\mathcal{C}_0(E) \subset \mathcal{C}_0(E)$ for any $t \in \mathbb{T} \setminus \{0\}$;

(3) *strong Feller* if $P_t\mathcal{M}_b(E) \subset \mathcal{C}_b(E)$ for any $t \in \mathbb{T} \setminus \{0\}$.

Remark 2.1.3. It should be noted that, under some technical conditions (including separability and local compactness of a state space), one may construct a Markov process with a given semigroup; see e.g. Theorem 3 in Section I.6 of Gikhman and Skorokhod (1975). In this case, the regularity properties of the semigroup imply certain useful properties of the Markov process. In particular, if the semigroup is $\mathcal{C}_b$-Feller and the corresponding process is right-continuous, then it is also strong Markov; see e.g. Theorem 3.10 in Dynkin (1965) for details. $\Diamond$

Although the Feller properties are defined for the semigroup, in the following, we identify the semigroup with the standard Markov process X. In particular, for brevity, we say that X is $\mathcal{C}_b$-Feller-Markov, $\mathcal{C}_0$-Feller-Markov or strong Feller-Markov if its transition semigroup satisfies the respective condition from Definition 2.1.2.

Let us now illustrate the properties from Definition 2.1.2 by a series of examples. First, in the following simple example we show that if the distribution of X is absolutely continuous with a continuous density, then the process is strong Feller.

Example 2.1.4. Suppose that, for any $t \in \mathbb{T} \setminus \{0\}$, we get

$$\mathbb{P}_x[X_t \in A] = \int_A p_t(x,y)\nu_t(dy), \quad A \in \mathcal{E}$$

for some probability measure ν_t on $(E, \mathcal{E})$ and a bounded density p_t which is continuous with respect to x variable. Then, using Lebesgue's dominated convergence theorem, we get that the process X is strong Feller. ♦

Next, note that, directly from Definition 2.1.2, we get that the strong Feller process is also $\mathcal{C}_b$-Feller. As we state in Proposition 2.1.5, a similar property holds for $\mathcal{C}_0$-Feller processes. For the proof, see Corollary 2.2 in Palczewski and Stettner (2010).

Proposition 2.1.5. *If X is a standard $\mathcal{C}_0$-Feller-Markov process, then X is also standard $\mathcal{C}_b$-Feller-Markov.*

Finally, in the following simple example we show that the implication of Proposition 2.1.5 cannot be reversed. However, it should be noted that if the state space E is compact, the notions of $\mathcal{C}_b$-Feller and $\mathcal{C}_0$-Feller processes coincide; this follows directly from the fact that, for a compact E, we get $\mathcal{C}_0(E) = \mathcal{C}_b(E)$.

Example 2.1.6. Let $\mathbb{T} := \mathbb{N}$ and $E := \mathbb{R}_+$. Let $\alpha \in [0,1]$ and $(X_n)_{n\in\mathbb{N}}$ be a discrete time Markov process with the transition probability

$$\mathbb{P}_x[X_1 = 0] = \alpha, \, \mathbb{P}_x[X_1 = x+1] = 1-\alpha, \quad x \in E.$$

Thus, for any $h \in \mathcal{M}_b(E)$, we get

$$\mathcal{P}_1 h(x) = \alpha h(0) + (1-\alpha)h(x+1), \quad x \in E.$$

Thus, the process X is $\mathcal{C}_b$-Feller, but it is not $\mathcal{C}_0$-Feller. ♦

Markov processes with the $\mathcal{C}_0$-Feller property are commonly used tools in the stochastic control theory. This setting includes Lévy processes and solutions to stochastic differential equations driven by Lévy processes; for details see Theorem 3.1.9 and Theorem 6.7.2 in Applebaum (2009), and Example 1.2.1 in Arapostathis et al. (2012). Also, $\mathcal{C}_0$-Feller-Markov processes are characterised by several useful regularity properties. In the following proposition, we collect two of them. For the proof, see Proposition 2.1 in Palczewski and Stettner (2010) and Proposition 6.4 in Basu and Stettner (2020).

Proposition 2.1.7. *Let X be a continuous time standard $\mathcal{C}_0$-Feller-Markov process. Then:*

(1) *For any compact set $\Gamma \subseteq E$ and $t_0 > 0$, we get*

$$\lim_{r\to\infty} \sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t_0]} \rho(X_s, x) \geq r \right] = 0.$$

(2) *For any compact set $\Gamma \subseteq E$ and $r_0 > 0$, we get*

$$\lim_{t\to 0} \sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t]} \rho(X_s, x) \geq r_0 \right] = 0.$$

The intuitive meaning of Proposition 2.1.7 is as follows: from (1), given some time horizon t_0, we may find $r > 0$ big enough such that with high probability, up to the time t_0, the process stays inside the ball with radius r. Similarly, from (2), given some radius r_0, we may find a time horizon t small enough such that with high probability, up to the time t, the process stays inside the ball with radius r_0. Also, in both cases, the estimates are uniform with respect to a starting point from some compact set.

Let us now introduce the second semigroup associated with a continuous time Markov process X. We fix some function $f \in \mathcal{C}_b(E)$ and define

$$\mathcal{P}_t^f h(x) := \mathbb{E}_x \left[e^{\int_0^t f(X_s)ds} h(X_t) \right], \quad t \geq 0,\, h \in \mathcal{M}_b(E),\, x \in E. \tag{2.1.5}$$

This semigroup could be associated with discounted problems or Markov processes with creation/killing. It can be shown that the $\mathcal{C}_b$-Feller property of $(\mathcal{P}_t)_{t\in\mathbb{R}_+}$ transfers into the similar property of $(\mathcal{P}_t^f)_{t\in\mathbb{R}_+}$. For the proof, see Lemma 4 in Section II.5 of Gikhman and Skorokhod (1975).

Proposition 2.1.8. *If X is a continuous time standard $\mathcal{C}_b$-Feller-Markov process, then, for any $t \in \mathbb{R}_+$, we get $\mathcal{P}_t^f \mathcal{C}_b(E) \subset \mathcal{C}_b(E)$.*

2.2 Impulse control strategies

In this section, we review the main points of the construction of a process corresponding to an impulse control strategy. For brevity, we do not include the proofs. An exhaustive discussion with detailed arguments can be found in Chapter V in Robin (1978); see also Section 2 in Stettner (1982), Section 54 in Davis (1993), and Appendix in Christensen (2014).

Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a continuous time standard Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. For transparency, we assume that the process is defined on the canonical space. In particular, we have that Ω is a family of càdlàg functions on $[0,\infty)$ with values in E and $X_t(\omega) = \omega(t)$, $t \geq 0$, $\omega \in \Omega$.

Before we provide details of the construction, let us outline the main idea. We wish to control the process X by impulses. An impulse control strategy $V := (\tau_i, \xi_i)_{i=1}^{\infty}$ consists of an increasing sequence of impulse times (τ_i) and a sequence of adapted after-impulse states (ξ_i). The controlled process starts at $x \in E$ and follows its uncontrolled dynamics up to τ_1. At this point, it is shifted to ξ_1, where it starts its (uncontrolled) evolution again, etc. The behaviour of the controlled process is captured by a new measure $\mathbb{P}_{(x,V)}$.

The main difficulty of the construction could be associated with the fact that we consider a discontinuous (càdlàg) uncontrolled Markov processes. Thus, it is possible that the jump and impulse times coincide and, in the construction, one needs to take into account before-jump, after-jump, and after-impulse states, which, in theory, are associated with the same time point. In fact, if the trajectories of the uncontrolled process are continuous, the construction could be simplified (up to some degree); see Helmes et al. (2019) for a detailed discussion.[1]

To formally introduce a controlled process, we define a new filtered measurable space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}})$ constructed with the help of the countable product of $(\Omega, \mathcal{F}, \mathbb{F})$. Thus, we set $\widehat{\Omega} := \prod_{k=1}^{\infty} \Omega$, $\widehat{\mathcal{F}} := \bigotimes_{k=1}^{\infty} \mathcal{F}$ and $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{t\geq 0}$ with $\widehat{\mathcal{F}}_t := \bigotimes_{k=1}^{\infty} \mathcal{F}_t$, $t \geq 0$, where $\prod$ denotes the Cartesian product and $\bigotimes$ denotes the product σ-field. On this space, we consider the coordinate process $\widehat{X} = (\widehat{X}_t)_{t\geq 0}$ defined, for any $t \geq 0$ and $\widehat{\omega} = (\omega_1, \omega_2, \ldots) \in \widehat{\Omega}$, by

$$\widehat{X}_t(\widehat{\omega}) := (X_t^1(\omega_1), X_t^2(\omega_2), \ldots) := (\omega_1(t), \omega_2(t), \ldots). \tag{2.2.1}$$

[1] The technical difficulties related to the formal construction of the controlled process were nicely summarised by M. Davis: "The formal probabilistic apparatus necessary to describe the above situation precisely, which we give next, is unfortunately rather cumbersome. This seems inevitable, but - fortunately - the details of the construction are only occasionally used explicitly in the ensuing developments, as will be seen below"; see Section 54 in Davis (1993).

Now, we give a more specific comment on an impulse control strategy $V := (\tau_i, \xi_i)_{i=1}^{\infty}$. For any $i \in \mathbb{N}_*$, we assume that τ_i is a stopping time with respect to the filtration $(\bigotimes_{k=1}^{i} \mathcal{F}_t \otimes \bigotimes_{k=i+1}^{\infty} \{\emptyset, \Omega\})$, $t \geq 0$, and with values in $\mathbb{R}_+ \cup \{+\infty\}$. Also, we assume that $\tau_i \leq \tau_{i+1}$, $i \in \mathbb{N}_*$. Next, we assume that ξ_i is a random variable taking values in E (or a fixed subset of E) and is measurable with respect to $\bigotimes_{k=1}^{i} \mathcal{F}_{\tau_i} \otimes \bigotimes_{k=i+1}^{\infty} \{\emptyset, \Omega\}$. In particular, we get that τ_i and ξ_i depend only on the first i coordinates of $\widehat{X}$. With a strategy V satisfying these assumptions we associate a controlled process $Y = (Y_t)_{t\geq 0}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}})$ given by

$$Y_t(\widehat{\omega}) := X_t^i(\widehat{\omega}) = \omega_i(t), \quad \widehat{\omega} = (\omega_1, \omega_2, \ldots) \in \widehat{\Omega},\ t \in [\tau_{i-1}(\widehat{\omega}), \tau_i(\widehat{\omega})),$$

with the convention $\tau_0 \equiv 0$. Also, to simplify the notation, by $Y_{\tau_i^-} := X^i_{\tau_i}$, $i \in \mathbb{N}_*$, we denote the state of the process right before the ith impulse (yet, possibly, after the jump).

The family of impulse control strategies satisfying the assumptions stated in the previous paragraphs is denoted by $\mathbb{V}$. By $\mathbb{V}^m \subset \mathbb{V}$, $m \in \mathbb{N}$, we denote the family of impulse control strategies with impulse times restricted to the time-grid $\{0, \delta_m, 2\delta_m, \ldots\} \cup \{+\infty\}$, where $\delta_m := (1/2)^m$. Next, for any $n \in \mathbb{N}$, by $\mathbb{V}_n \subset \mathbb{V}$ we denote the family of impulse control strategies with at most n impulses (i.e. with $\tau_{n+1} \equiv +\infty$). Also, for any $n, m \in \mathbb{N}$, we define $\mathbb{V}_n^m := \mathbb{V}_n \cap \mathbb{V}^m$.

With a strategy $V = (\tau_i, \xi_i)_{i=1}^{\infty} \in \mathbb{V}$ and a starting point $x \in E$ we associate a new probability measure $\mathbb{P}_{(x,V)}$ such that

$$\mathbb{P}_{(x,V)}[X_t^1 \in A] = \mathbb{P}_x[X_t^1 \in A], \quad t \in [0, \tau_1),\ A \in \mathcal{F},$$

and for any $i \in \mathbb{N}_*$, $t \geq 0$, $\widehat{\mathcal{F}}^i_{\tau_i} := \bigotimes_{k=1}^{i} \mathcal{F}_{\tau_i} \otimes \mathcal{F}_{\tau_i^-} \otimes \bigotimes_{k=i+2}^{\infty} \{\emptyset, \Omega\}$, and $A_1, \ldots, A_{i+1} \in \mathcal{F}$, we get

$$\begin{aligned}\mathbb{P}_{(x,V)}[X^1_{\tau_i+t} \in A_1, \ldots, X^i_{\tau_i+t} \in A_i, X^{i+1}_{\tau_i+t} \in A_{i+1} | \widehat{\mathcal{F}}^i_{\tau_i}]& \\ = \delta_{y_{X^1_{\tau_1}}}(A_1) \cdot \ldots \cdot \delta_{y_{X^i_{\tau_i}}}(A_i) \mathbb{P}_{\xi_i}[X_t^{i+1} \in A_{i+1}] \text{ on } \{\tau_i + t < \tau_{i+1}\},&\end{aligned}$$

where δ_z is the Dirac measure at z and y_a denotes the constant trajectory $y_a(t) := a$, $a \in E$, $t \geq 0$. In particular, under $\mathbb{P}_{(x,V)}$, we get $Y_{\tau_i} = \xi_i$ and the controlled process between the consecutive impulse times is Markov with the original (uncontrolled) dynamics. In the following, we use $\mathbb{E}_{(x,V)}$ to denote the expectation operator corresponding to $\mathbb{P}_{(x,V)}$, $x \in E$, $V \in \mathbb{V}$.

It should be noted that the uncontrolled process X could be naturally embedded in the controlled framework. In particular, we may identify X with some coordinate of the process $\widehat{X}$ given by (2.2.1) or with the result of the use of the *no impulse* strategy, i.e. $V = (\tau_i, \xi_i)_{i=1}^{\infty}$ with $\tau_1 \equiv +\infty$. Nevertheless, with a slight abuse of notation, we use $\mathbb{P}_x$, $x \in E$, to denote the dynamics of the uncontrolled process.

2.3 Entropic utility measure

In this section, we discuss the properties of the entropic utility measure and the associated risk-sensitive criterion. We present some basic mathematical properties and provide their economic interpretation. Also, in the end, we comment on the standardised version of the risk-sensitive control problems that are considered in this thesis. This section is based mainly on Bielecki and Pliska (2003); see also Whittle (1990).

2.3.1 Generic properties

In this section, for simplicity, we fix some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also, we use L^∞ to denote the family of essentially bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. This space provides a framework for the definitions and properties discussed in this section. It should be noted that most of them could be extended to the unbounded case, assuming suitable regularity and integrability conditions. Nevertheless, for transparency, we focus on L^∞.

Let us now present the definition of the entropic utility measure.

Definition 2.3.1. The entropic utility of $Z \in L^\infty$ with the risk-aversion parameter $\gamma \in \mathbb{R}$ is given by

$$J^\gamma(Z) := \begin{cases} \frac{1}{\gamma} \ln \mathbb{E}\left[\exp(\gamma Z)\right], & \gamma \neq 0, \\ \mathbb{E}\left[Z\right], & \gamma = 0. \end{cases} \tag{2.3.1}$$

Now, we provide a series of comments on this definition. First, note that in the economic context, random variable Z corresponds to some (random) outcome of an investment project (e.g. pay-off or rate of return corresponding to some investment strategy) and $J^\gamma(Z)$ could be used to measure an investor's satisfaction associated with Z. In this case, the parameter $\gamma \in \mathbb{R}$ corresponds to a decision-maker risk-aversion. The case $\gamma < 0$ (respectively $\gamma > 0$) describes the preferences of the *risk-averse* (respectively *risk-seeking*) investor, while $\gamma = 0$ corresponds to the *risk-neutral* preferences. Also, note that $J^\gamma(Z)$ for $\gamma < 0$ could be seen as the certainty equivalent for the exponential utility function $x \mapsto U^\gamma(x) := \frac{1}{\gamma} e^{\gamma x}$, $\gamma < 0$. Indeed, directly from the definition, we get $U^\gamma(J^\gamma(Z)) = \mathbb{E}[U^\gamma(Z)]$. Thus, from the economic point of view, $J^\gamma(Z)$ could be linked to a constant (deterministic) pay-off, which utility is equal to the expected utility of Z.

Second, in the optimisation context, J^γ could be used as an optimality criterion quantifying the performance of some control affecting Z. In this case, a decision-maker tries to find a strategy such that the entropic utility of the resulting random variable Z is maximised. In the stochastic control literature, the main focus is set on the case $\gamma < 0$ as this seems to provide a reasonable explanation of investors' behaviour. Typically, the functional J^γ with $\gamma < 0$ is referred to as the *risk-sensitive criterion*. We refer to Howard and Matheson (1972) for a seminal contribution on the use of J^γ in the optimisation context.

Third, as we show now, J^γ could be seen as a non-linear extension of the classic mean-variance optimisation framework. To see this, recall that the moment and cumulant generating functions of $Z \in L^\infty$ are given by $t \mapsto M_Z(t) := \mathbb{E}[e^{tZ}]$ and $t \mapsto C_Z(t) := \ln M_Z(t)$, respectively; see e.g. Section 9 in Billingsley (1995) for a comprehensive overview. Thus, directly from the definition, we get $J^\gamma(Z) = \frac{1}{\gamma} C_Z(\gamma)$, $\gamma \neq 0$. Also, using the discussion following Equation (9.7) in Billingsley (1995), we get that, in some neighbourhood of 0, the map $t \mapsto C_Z(t)$ could be expanded in the Taylor series of the form

$$C_Z(t) = \sum_{k=1}^{\infty} \frac{c_k(Z)}{k!} t^k,$$

where $c_k(Z)$, $k \in \mathbb{N}_*$, are the cumulants of the random variable Z. In particular, we get $c_1(Z) = \mathbb{E}[Z]$, $c_2(Z) = \mathrm{Var}[Z]$, and for small γ, we get

$$J^\gamma(Z) = \mathbb{E}[Z] + \frac{\gamma}{2} \mathrm{Var}[Z] + O(\gamma^2). \tag{2.3.2}$$

This formula highlights the connection between the entropic utility measure and the linear-quadratic (mean-variance) approach of Markowitz (1952). In fact, optimisation of J^γ could be seen as maximisation of the expected return penalised for the risk associated with the investment project. However, in contrast to the Markowitz approach, where the risk is measured only by the variance, J^γ accounts also for the higher-order moments. This makes the outcomes smoother and more stable, e.g. when a non-Gaussian framework is considered. Also, note that, for a Gaussian random variable Z, we get $C_Z(t) = t\mathbb{E}[Z] + \frac{t^2}{2} \mathrm{Var}[Z]$, $t \in \mathbb{R}$. Thus, in (2.3.2) we get the equality for any $\gamma \in \mathbb{R}$ even without $O(\gamma^2)$ term.

Let us now summarise some basic mathematical properties of J^γ. For brevity, we focus on the risk-sensitive case. The properties follow directly from the definition and Jensen inequality; see e.g. Bielecki and Pliska (2003) for details.

Proposition 2.3.2. *Let $\gamma < 0$ and let J^γ be given by (2.3.1). Then:*

(1) *For any $Z \in L^\infty$, we get that the map $(-\infty, 0) \ni \gamma \mapsto J^\gamma(Z)$ is increasing. Also, we get $J^\gamma(Z) \leq \mathbb{E}[Z]$, $\gamma < 0$.*

(2) *For any $Z_1, Z_2 \in L^\infty$ such that $Z_1 \leq Z_2$, we get $J^\gamma(Z_1) \leq J^\gamma(Z_2)$.*

(3) *For any $Z \in L^\infty$ and $a \in \mathbb{R}$, we get $J^\gamma(Z + a) = J^\gamma(Z) + a$.*

(4) *For any $Z_1, Z_2 \in L^\infty$ and $\alpha \in [0, 1]$, we get*

$$J^\gamma(\alpha Z_1 + (1-\alpha)Z_2) \geq \alpha J^\gamma(Z_1) + (1-\alpha)J^\gamma(Z_2).$$

Let us now comment on the properties from Proposition 2.3.2. From (1), we get that the satisfaction of an investor using J^γ, $\gamma < 0$, is bounded from above by the satisfaction associated with the risk-neutral preferences. Next, (2) states the monotonicity J^γ. Economically speaking, an investor using J^γ as an investment criterion is more satisfied with the strategy resulting in the (almost surely) greater profit. Next, (3) is linked to the cash-invariance of J^γ. From the economic perspective, if one adds to the portfolio a units of cash, the resulting satisfaction (measured by J^γ) increases by a. Finally, (4) states the concavity of J^γ. For Z_1 and Z_2 corresponding to some investment portfolios (or their rate of returns), (4) says that the satisfaction of the combined portfolio $\alpha Z_1 + (1-\alpha)Z_2$ (with some rate a) is greater than the averaged satisfaction of the separate portfolios.

Properties (2)–(4) provides a link between the entropic utility measure and the theory of risk measures. More specifically, we get that $\rho^\gamma(Z) := -J^\gamma(Z)$, $\gamma < 0$, $Z \in L^\infty$, is a convex risk measure, i.e. a normalised, monotone, cash-invariant, and convex functional that can be used to quantify the risk associated with some portfolio; see e.g. Chapter 4 in Föllmer and Schied (2016) for details. In fact, it can be shown that $J^\gamma(Z)$ is essentially the only functional resulting in the risk measure that is characterised by these properties; see Kupper and Schachermayer (2009). In particular, we get that, in the dynamic context, the entropic utility measure is time-consistent, which facilitates the use of the dynamic programming principle; see e.g. Bielecki et al. (2015).

Let us now comment on the connection between J^γ and the entropy (information content) of the probability distribution; see Proposition 2.3.3. Before we state the proposition, we introduce some auxiliary notation. We use

$\mathbb{M}(\Omega, \mathcal{F})$ to denote the family of probability measures on $(\Omega, \mathcal{F})$. Also, for $\mathbb{Q} \in \mathbb{M}(\Omega, \mathcal{F})$, let $\mathbb{E}^{\mathbb{Q}}$ denote the expectation operator corresponding to $\mathbb{Q}$. Finally, for $\mathbb{Q} \in \mathbb{M}(\Omega, \mathcal{F})$, we define $H(\mathbb{Q}, \mathbb{P})$ as the Kullback-Leibler divergence (relative entropy) given by

$$H(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], & \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\mathbb{Q} \ll \mathbb{P}$ means that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$.

Now, we are ready to state Proposition 2.3.3. The proof can be found e.g. in Lemma 3.29 in Föllmer and Schied (2016); see also Lemma A.1 in Bartl (2019) for a discussion on the unbounded case.

Proposition 2.3.3. *Let $\gamma < 0$ and let J^γ be given by (2.3.1). Then, for any $Z \in L^\infty$, we get*

$$J^\gamma(Z) = \inf_{\mathbb{Q} \in \mathbb{M}(\Omega, \mathcal{F})} \left(\mathbb{E}^{\mathbb{Q}}(Z) - \frac{1}{\gamma} H(\mathbb{Q}, \mathbb{P})\right), \quad \gamma \neq 0. \tag{2.3.3}$$

Formula (2.3.3) could be seen as a variational representation of $J^\gamma(Z)$ and is linked to the Fenchel-Legendre transformation; see e.g. Section 3.2 in Föllmer and Schied (2016) for details. Also, it shows that maximisation of J^γ could be seen as a maximin problem with a regularisation term related to the relative entropy. We refer to Cherny and Maslov (2004) for further discussion on the connections between entropic utility measure and information theory.

2.3.2 Standardised control problems

In this section, we comment on the specific formulation of the problems (1.0.3) and (1.0.4), and their link to J^γ. We show that, for a fixed parameter $\gamma < 0$, these problems could be expressed in a simpler, standardised form.

Recall that the first class of problems considered in this thesis is associated with optimal stopping. More specifically, let $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ be a continuous time standard $\mathcal{C}_b$-Feller-Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Also, let $\gamma < 0$ and let J^γ_x, $x \in E$, be a version of (2.3.1) with $\mathbb{E}$ replaced by $\mathbb{E}_x$. In this framework, given some running cost function $\widehat{g}$ and terminal reward/cost function $\widehat{G}$, we consider problems of the form

$$\sup_{\tau \in \mathcal{T}_x} J^\gamma_x \left(\int_0^\tau \widehat{g}(X_s) ds + \widehat{G}(X_\tau)\right), \quad x \in E. \tag{2.3.4}$$

As we show now, for a fixed parameter $\gamma < 0$, this class of problems could be expressed in a simpler form. More specifically, setting $g(\cdot) := \gamma\widehat{g}(\cdot)$ and $G(\cdot) := \gamma\widehat{G}(\cdot)$, and recalling that $\gamma < 0$, we get that (2.3.4) is equivalent to

$$\inf_{\tau\in\mathcal{T}_x} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau)\right)\right], \quad x \in E. \tag{2.3.5}$$

Thus, we get an optimal stopping problem in the multiplicative form. More specific comments on (2.3.5) and its connection with the framework considered in the literature can be found in Section 3.1.

The second class of problems considered in this thesis is linked to the impulse control setting. For a continuous time standard Markov process X, $\gamma < 0$, and an impulse control strategy $V \in \mathbb{V}$, by $J^\gamma_{(x,V)}$ we denote the version of the entropic utility measure (1.0.2) corresponding to $\mathbb{E}_{(x,V)}$. In this context, we consider problems of the form

$$\sup_{V\in\mathbb{V}} \liminf_{T\to\infty} \frac{1}{T} J^\gamma_{(x,V)}\left(\int_0^T \widehat{f}(Y_s)ds + \sum_{i=1}^\infty 1_{\{\tau_i\leq T\}}\widehat{c}(Y_{\tau_i^-}, \xi_i)\right), \quad x \in E, \tag{2.3.6}$$

where $\widehat{f}$ and $\widehat{c}$ are running reward/cost function and shift-cost function, respectively. Setting $f(\cdot) := \gamma\widehat{f}(\cdot)$ and $c(\cdot,\text{-}) := \gamma\widehat{c}(\cdot,\text{-})$, by analogy to (2.3.5), we get that (2.3.6) is equivalent to

$$\inf_{V\in\mathbb{V}} \limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{E}_{(x,V)}\left[\exp\left(\int_0^T f(Y_s)ds + \sum_{i=1}^\infty 1_{\{\tau_i\leq T\}}c(Y_{\tau_i^-}, \xi_i)\right)\right], \; x \in E. \tag{2.3.7}$$

More specific comments on this type of problems can be found in Section 4.1.

Problems (2.3.5) and (2.3.7) could be seen as the standardised versions of the risk-sensitive functionals, where the risk-aversion parameter γ is not used directly (apart from its sign). It should be noted that, in some cases, the magnitude of γ is important, e.g. when the asymptotic behaviour of the optimal value of some control problem is considered; see e.g. Menaldi and Robin (2005). Nevertheless, the framework of this thesis allows us to consider any $\gamma < 0$. Consequently, from now on we focus on the normalised problems of the form (2.3.5) and (2.3.7) keeping in mind that they correspond to generic problems (2.3.4) and (2.3.6).

Chapter 3

Optimal stopping problems

In this chapter, we consider several forms of risk-sensitive optimal stopping problems for Feller–Markov processes. Under suitable assumptions, we show specific regularity properties of the optimal stopping value functions, including continuity with respect to a starting point and various approximation schemes. Also, we link the value functions to the specific form of the Bellman equation and show that, in the unbounded framework, this equation may have multiple solutions. Our analysis covers finite and infinite time horizons as well as discrete and continuous time frameworks.

The structure of this chapter is as follows. In Section 3.1, we formally introduce the problem and discuss the set of assumptions. Next, in Section 3.2, we study discrete time optimal stopping problems. The main contribution of this part is Theorem 3.2.6, where we link the discrete time Bellman equation to the limits of the suitable finite time horizon stopping problems. In Section 3.3, we study finite horizon continuous time optimal stopping problems. In Theorem 3.3.2, we show that the corresponding value functions are continuous with respect to space and time variables. This is used in Section 3.4, where we give a characterisation of solutions to the continuous time Bellman equation; see Theorem 3.4.3 for details. Also, in Theorem 3.4.7, we show a condition for the uniqueness of a solution to this equation. In Section 3.5, we show various approximation results for the optimal stopping value functions. For transparency, in Section A.1 and Section A.2 in Appendix A, we collect some standard results for the optimal stopping problems that are used in this chapter. Also, our results are illustrated by the examples presented in Chapter 5. In particular, in Example 5.3.1 and Example 5.3.5 we show explicit formulae for distinct solutions to the Bellman equation.

The results presented in this chapter are based mainly on Jelito et al.

(2021) and Jelito and Stettner (2022). In particular, in the first paper we showed some regularity properties of the optimal stopping problems in the bounded framework. In the second paper we extended our analysis to the unbounded setting and show that the corresponding Bellman equation may admit multiple solutions. Also, note that in this chapter we provide some more detailed comments on the arguments presented in the papers. In particular, this applies to the proofs of Theorem 3.3.2 and Theorem 3.3.4.

3.1 Problem statement and assumptions

In this section, we state the main problem and introduce the notation and assumptions used throughout this chapter. We focus on the continuous time optimal stopping framework; the necessary modifications for the discrete time case are discussed at the beginning of Section 3.2.

The main focus of this chapter is set on the infinite time horizon optimal stopping problems. Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a continuous time standard $\mathcal{C}_b$-Feller–Markov process on a filtered measurable space $(\Omega, \mathcal{F}, \mathbb{F})$ with values in a state space $(E, \mathcal{E})$. We consider

$$\underline{w}(x) := \inf_{\tau\in\mathcal{T}_x} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau)\right)\right], \quad x \in E, \tag{3.1.1}$$

$$\overline{w}(x) := \inf_{\tau\in\mathcal{T}_{x,b}} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau)\right)\right], \quad x \in E, \tag{3.1.2}$$

where $g \in \mathcal{C}_b^+(E)$ and $G \in \mathcal{C}^+(E)$ are a *running cost function* and a *terminal reward/cost function*, respectively.

The map $\underline{w}$ describes the value of a standard risk-sensitive optimal stopping problem; cf. (2.3.5). The map $\overline{w}$ may be seen as a version of $\underline{w}$, when a decision-maker is allowed to choose only bounded stopping times. Note that directly from the fact $\mathcal{T}_{x,b} \subset \mathcal{T}_x$, $x \in E$, we get $\underline{w}(x) \leq \overline{w}(x)$, $x \in E$. In this chapter, we show that both functions $\underline{w}$ and $\overline{w}$ are solutions to the associated optimal stopping Bellman equation. In fact, we show that $\underline{w}$ and $\overline{w}$ are minimal and maximal solutions to this equation, respectively, and, in general, we do not have the equality between $\underline{w}$ and $\overline{w}$; see Theorem 3.4.3 for details. It should be noted that the non-uniqueness of a solution to the Bellman equation could be associated with the unboundedness of G. In fact, in Theorem 3.4.11 we show that when G is bounded, the Bellman equation admits a unique solution. A more general condition for this property can be found in Theorem 3.4.7.

To obtain a proper regularity of the problems, throughout this chapter we make the following assumptions:

($\mathcal{A}1$) (Cost functions constraints). We have $g \in \mathcal{C}_b^+(E)$ and $G \in \mathcal{C}^+(E)$. Also, the function g is bounded away from zero, i.e. for some $c > 0$ we have $g(\cdot) \geq c > 0$.

($\mathcal{A}2$) (Integrability). For any $x \in E$ and $T \geq 0$, we have

$$\mathbb{E}_x \left[\sup_{t\in[0,T]} e^{G(X_t)} \right] < \infty. \tag{3.1.3}$$

($\mathcal{A}3$) (Continuity). For any $T \geq 0$ and $h \in \mathcal{C}^+(E)$ satisfying $h(x) \leq G(x)$, $x \in E$, it holds that the map

$$x \mapsto \mathbb{E}_x \left[\exp\left(\int_0^T f(X_s)ds + h(X_T) \right) \right] \tag{3.1.4}$$

is continuous.

($\mathcal{A}4$) (Distance control). For any compact set $\Gamma \subset E$, $t_0 > 0$, and $r_0 > 0$, we have

$$\lim_{r\to\infty} M_\Gamma(t_0, r) = 0, \qquad \lim_{t\to 0} M_\Gamma(t, r_0) = 0, \tag{3.1.5}$$

where $M_\Gamma(t, r) := \sup_{x\in\Gamma} \mathbb{P}_x[\sup_{s\in[0,t]} \rho(X_s, x) \geq r]$, $t, r > 0$.

Let us now comment on these assumptions.

Assumption ($\mathcal{A}1$) imposes several technical regularity properties on the cost functions. First, note that the non-negativity assumption for G is merely a technical normalisation. Indeed, for a generic $\widetilde{G} \in \mathcal{C}(E)$ which is bounded from below, we may subtract the quantity $\inf_{y\in E} \widetilde{G}(y)$ from both sides of (3.1.1) and (3.1.2), and set $G(\cdot) := \widetilde{G}(\cdot) - \inf_{y\in E} \widetilde{G}(y) \in \mathcal{C}^+(E)$. Second, the assumption $g(\cdot) \geq c > 0$ implies that allowing for infinite-valued stopping times in the definition of $\underline{w}$ does not improve the optimal value; see Proposition 3.1.1 for details. Note that this condition is not needed in the finite time horizon framework. Also, in Chapter 4 we show how to relax this assumption with the help of the Multiplicative Poisson Equation; see Proposition 4.3.2 and the following discussion for details.

Assumption ($\mathcal{A}2$) guarantees integrability in the finite time horizon setting and is a standard condition in the optimal stopping literature; see e.g. condition (D.29) in Karatzas and Shreve (1998*b*).

Assumption ($\mathcal{A}3$) gives the continuity of the specific exponential semigroup for all unbounded functions $h \in \mathcal{C}^+(E)$ that are majorised by G. Note that from the $\mathcal{C}_b$-Feller property, Proposition 2.1.8, and the monotone convergence theorem, we get that the map from (3.1.4) is lower semi-continuous for any $T \geq 0$ and $h \in \mathcal{C}^+(E)$. Thus, in fact, in Assumption ($\mathcal{A}3$) we impose the upper semi-continuity of the map from (3.1.4).

Assumption ($\mathcal{A}4$) facilitates distance control of the Markov process. The property $\lim_{r\to\infty} M_\Gamma(t_0, r) = 0$ implies that, for any fixed time horizon $t_0 > 0$, the process cannot get too far from the starting point, while the property $\lim_{t\to 0} M_\Gamma(t, r_0) = 0$ indicates that the process does not exhibit frequent sudden jumps on small time intervals. Both properties are satisfied if the underlying process is $\mathcal{C}_0$-Feller; see Proposition 2.1.7.

It should be noted that Assumptions ($\mathcal{A}2$)–($\mathcal{A}3$) are automatically satisfied if G is bounded. In particular, Assumption ($\mathcal{A}3$) follows from the $\mathcal{C}_b$-Feller property and Proposition 2.1.8. Further comments on these assumptions can be found in Section 5.1. More specifically, in that section we show that Assumptions ($\mathcal{A}2$)–($\mathcal{A}3$) can be deduced from a more general condition related to the integrability of tails of a suitable random variable; see Lemma 5.1.1 for details.

Now, let us comment on the link between our framework and settings considered in the classic optimal stopping literature. In principle, by setting $Z(t) := e^{-\int_0^t r(X_s)ds} h\left(X_t\right)$, $t \geq 0$, with $r(\cdot) := -g(\cdot)$ and $h(\cdot) := e^{G(\cdot)}$, the optimal stopping problems associated with (3.1.1) and (3.1.2) might be embedded into the classic (discounted) optimal stopping framework $\inf_\tau \mathbb{E}_x\left[Z(\tau)\right]$; see e.g. Peskir and Shiryaev (2006). Nevertheless, in the classic approach, the map r corresponds to the discount factor and usually it is assumed to be strictly positive; see e.g. Robin (1978); Pham (2009), and references therein. However, in our setting r is strictly negative; cf. Assumption ($\mathcal{A}1$). Consequently, the standard tools used in the discounted optimal stopping cannot be directly applied. In particular, we do not get the integrability condition for the value process $\mathbb{E}_x[\sup_{t\geq 0}|Z_t|] < \infty$, $x \in E$, which is a standard tool in the classic framework; see e.g. condition (2.2.1) in Peskir and Shiryaev (2006).

Problems associated with (3.1.1) and (3.1.2) could also be analysed using state-space transformation. More specifically, we may introduce a new Markov process $\widetilde{X}_t := (\int_0^t g(X_s)ds, X_t)$, $t \geq 0$, defined on an appropriately enlarged state space and consider the problem $\inf_\tau \mathbb{E}[h(\widetilde{X}_\tau)]$, where $h(x, y) = e^{x+G(y)}$; see Peskir and Shiryaev (2006) or Getoor (1988) for a related concept of a Markov process with creation. However, this approach does not use the special structure of (3.1.1) and (3.1.2), and is associated with certain technical difficulties; see Section 6 in Peskir and Shiryaev (2006) for further discussion. Thus, to streamline the analysis, we decided to use a direct approach which also facilitates better presentation of the results. Still, note that by exploiting the state-space transformation our results could be used to solve time-inhomogeneous (time-dependent) problems; see Theorem 3.3.4 for details.

Let us now comment on the specific forms of the maps $\underline{w}$ and $\overline{w}$ given by (3.1.1) and (3.1.2). In the context of infinite time horizon optimal stopping, one needs to properly define the value of the process stopped at infinity. Usually, allowing for infinite-valued stopping times facilitates the existence of an optimal stopping rule. More specifically, let $(Z(t))$, $t \geq 0$, be a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, describing the associated pay-off. In most frameworks, the value of the process stopped at infinity is defined as $Z(\infty) := \liminf_{T\to\infty} Z(T)$, which results in the optimal stopping problem of the form

$$\inf_{\tau\in\mathcal{T}} \mathbb{E}\left[Z(\tau)1_{\{\tau<\infty\}} + \liminf_{T\to\infty} Z(T)1_{\{\tau=\infty\}}\right]; \tag{3.1.6}$$

see e.g. Equation (2) in Fakeev (1970) or Equation (2.5) in Shiryaev (1978). If the process $(Z(t))$, $t \geq 0$, is quasi left-continuous, Problem (3.1.6) could be written in a compact form

$$\inf_{\tau\in\mathcal{T}} \mathbb{E}\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right]. \tag{3.1.7}$$

Interchanging the limit and the expectation operator, we get an alternative formulation

$$\inf_{\tau\in\mathcal{T}} \liminf_{T\to\infty} \mathbb{E}\left[Z(\tau\wedge T)\right]; \tag{3.1.8}$$

see e.g. Equation (1) in Palczewski and Stettner (2014). In Proposition 3.1.1 we use the idea of (3.1.7) and (3.1.8) to give the alternative definitions of the maps $\underline{w}$ and $\overline{w}$. Also, we show that, in our framework, allowing for stopping times that take the infinite value with positive probability does not improve the optimal values of these problems. To simplify the notation, we set

$$Z(t) := \exp\left(\int_0^t g(X_s)ds + G(X_t)\right), \quad t \geq 0. \tag{3.1.9}$$

Proposition 3.1.1. *Let the maps $\underline{w}$ and $\overline{w}$ be given by (3.1.1) and (3.1.2), respectively. Also, let the process Z be given by (3.1.9). Then:*

(1) *We get*

$$\begin{aligned}\underline{w}(x) &= \inf_{\tau\in\mathcal{T}_x} \ln \mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right]\\ &= \inf_{\tau\in\mathcal{T}} \ln \mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right], \quad x \in E.\end{aligned} \tag{3.1.10}$$

(2) *We get*

$$\begin{aligned}\overline{w}(x) &= \inf_{\tau\in\mathcal{T}_x}\liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau\wedge T)\right]\\ &= \inf_{\tau\in\mathcal{T}}\liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau\wedge T)\right], \quad x\in E.\end{aligned} \tag{3.1.11}$$

Proof. For transparency, we prove the claims point by point.

Proof of (1). Note that, from the quasi left-continuity of X and the continuity of G and g, we get that the process Z is quasi left-continuous. Thus, for any $\tau\in\mathcal{T}$ and $x\in E$, we get

$$\liminf_{T\to\infty} Z(\tau\wedge T) = Z(\tau)1_{\{\tau<\infty\}} + \liminf_{T\to\infty} Z(T)1_{\{\tau=\infty\}}.$$

In particular, for $\tau\in\mathcal{T}_x$, we get $Z(\tau)=\liminf_{T\to\infty}Z(\tau\wedge T)$, which shows the first equality in (3.1.10). For the second equality, note that, using Assumption ($\mathcal{A}1$), we get $\liminf_{T\to\infty}Z(T)\equiv\infty$. Thus, for any $x\in E$, we get that any stopping time $\tau\in\mathcal{T}$ such that $\mathbb{P}_x[\tau=\infty]>0$ cannot be optimal for the problem $\inf_{\tau\in\mathcal{T}}\ln\mathbb{E}_x\left[\liminf_{T\to\infty}Z(\tau\wedge T)\right]$. Consequently, we get

$$\inf_{\tau\in\mathcal{T}}\ln\mathbb{E}_x\left[\liminf_{T\to\infty}Z(\tau\wedge T)\right] = \inf_{\tau\in\mathcal{T}_x}\ln\mathbb{E}_x\left[\liminf_{T\to\infty}Z(\tau\wedge T)\right] = \underline{w}(x),\ x\in E,$$

which concludes the proof of this point.

Proof of (2). First, note that, using the boundedness of $\tau\in\mathcal{T}_{x,b}$ and the fact that $\mathcal{T}_{x,b}\subset\mathcal{T}_x$, $x\in E$, we get

$$\begin{aligned}\overline{w}(x) &= \inf_{\tau\in\mathcal{T}_{x,b}}\liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau\wedge T)\right]\\ &\geq \inf_{\tau\in\mathcal{T}_x}\liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau\wedge T)\right], \quad x\in E.\end{aligned} \tag{3.1.12}$$

Second, let $x\in E$, $\varepsilon>0$, and $\tau_\varepsilon\in\mathcal{T}_x$ be an ε-optimal stopping time for $\inf_{\tau\in\mathcal{T}_x}\liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau\wedge T)\right]$. Also, let $(T_n)\subset\mathbb{R}_+$ be a sequence such that $T_n\to\infty$ as $n\to\infty$ and

$$\lim_{n\to\infty}\ln\mathbb{E}_x\left[Z(\tau_\varepsilon\wedge T_n)\right] = \liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau_\varepsilon\wedge T)\right].$$

Then, noting that $\tau_\varepsilon\wedge T_n\in\mathcal{T}_{x,b}$, $n\in\mathbb{N}$, we get

$$\begin{aligned}\overline{w}(x) &\leq \lim_{n\to\infty}\ln\mathbb{E}_x\left[Z(\tau_\varepsilon\wedge T_n)\right]\\ &= \liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau_\varepsilon\wedge T)\right]\\ &\leq \inf_{\tau\in\mathcal{T}_x}\liminf_{T\to\infty}\ln\mathbb{E}_x\left[Z(\tau\wedge T)\right]+\varepsilon.\end{aligned}$$

Thus, letting $\varepsilon \to 0$, we get $\overline{w}(x) \leq \inf_{\tau\in\mathcal{T}_x} \liminf_{T\to\infty} \ln \mathbb{E}_x\left[Z(\tau\wedge T)\right]$, $x \in E$. Combining this with (3.1.12), we get the first equality in (3.1.11).

Finally, let $x \in E$ and $\tau \in \mathcal{T}$ be such that $\mathbb{P}_x[\tau = \infty] > 0$. Then, using Assumption $\mathcal{A}1$, on the event $\{\tau = \infty\}$, we get $\liminf_{T\to\infty} Z(\tau\wedge T) = \infty$ and, using Fatou's lemma, we get

$$\infty = \mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right] \leq \liminf_{T\to\infty} \mathbb{E}_x\left[Z(\tau\wedge T)\right].$$

Thus, any stopping time $\tau \in \mathcal{T}$ such that $\mathbb{P}_x[\tau = \infty] > 0$ cannot be optimal for the problem $\inf_{\tau\in\mathcal{T}} \liminf_{T\to\infty} \ln \mathbb{E}_x\left[Z(\tau\wedge T)\right]$, $x \in E$. Consequently, we get

$$\inf_{\tau\in\mathcal{T}} \liminf_{T\to\infty} \ln \mathbb{E}_x\left[Z(\tau\wedge T)\right] = \inf_{\tau\in\mathcal{T}_x} \liminf_{T\to\infty} \ln \mathbb{E}_x\left[Z(\tau\wedge T)\right] = \overline{w}(x),\ x \in E,$$

which concludes the proof of this point. $\square$

Formulations (3.1.7) and (3.1.8) provide better understanding of the structure of (3.1.1) and (3.1.2). More specifically, using Proposition 3.1.1 and Fatou's lemma, for any $x \in E$, we get

$$\underline{w}(x) = \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right] \leq \inf_{\tau\in\mathcal{T}} \liminf_{T\to\infty} \mathbb{E}_x\left[Z(\tau\wedge T)\right] = \overline{w}(x).$$

Based on this formula we may deduce when the identity $\underline{w} \equiv \overline{w}$ holds. More specifically, we get that if, for a suitable class of stopping times τ and $x \in E$, we get

$$\mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right] = \liminf_{T\to\infty} \mathbb{E}_x\left[Z(\tau\wedge T)\right]$$

then we get $\underline{w}(x) = \overline{w}(x)$. In the following lemma we show a useful characterisation of this condition.

Lemma 3.1.2. *Let Z be given by (3.1.9). Also, let $x \in E$ and $\tau \in \mathcal{T}_x$ be a stopping time satisfying $\mathbb{E}_x\left[Z(\tau)\right] < \infty$. Then, the following are equivalent:*

(1) *We get*

$$\liminf_{T\to\infty} \mathbb{E}_x\left[Z(\tau\wedge T)\right] = \mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right].$$

(2) *The family $\{Z_{\tau\wedge T}\}$, $T \geq 0$, is $\mathbb{P}_x$-uniformly integrable, i.e.*

$$\lim_{n\to\infty} \sup_{T\geq 0} \mathbb{E}_x\left[1_{\{Z(\tau\wedge T)\geq n\}} Z(\tau\wedge T)\right] = 0.$$

(3) *We get*
$$\liminf_{T\to\infty} \mathbb{E}_x\left[1_{\{\tau>T\}}Z(T)\right] = 0.$$

Proof. Note that the equivalence of (1) and (2) follows from the standard result; see e.g. Theorem 16.14 in Billingsley (1995) for details. Thus, it is enough to show that (1) is equivalent to (3). Using the identity

$$\mathbb{E}_x\left[Z(\tau\wedge T)\right] = \mathbb{E}_x\left[1_{\{\tau\le T\}}Z(\tau)\right] + \mathbb{E}_x\left[1_{\{\tau>T\}}Z(T)\right], \quad T\ge 0, \tag{3.1.13}$$

and noting that $1_{\{\tau\le T\}}Z(\tau)$ increases to $Z(\tau)$ as $T\to\infty$, by the monotone convergence theorem and quasi the left-continuity of Z, we get

$$\lim_{T\to\infty} \mathbb{E}_x\left[1_{\{\tau\le T\}}Z(\tau)\right] = \mathbb{E}_x\left[Z(\tau)\right] = \mathbb{E}_x\left[\lim_{T\to\infty} Z(\tau\wedge T)\right].$$

Thus, letting $T\to\infty$ in (3.1.13), we get

$$\liminf_{T\to\infty} \mathbb{E}_x\left[Z(\tau\wedge T)\right] = \mathbb{E}_x\left[\lim_{T\to\infty} Z(\tau\wedge T)\right] + \liminf_{T\to\infty} \mathbb{E}_x\left[1_{\{\tau>T\}}Z(T)\right].$$

Consequently, recalling that $\mathbb{E}_x\left[\lim_{T\to\infty} Z(\tau\wedge T)\right] = \mathbb{E}_x\left[Z(\tau)\right] < \infty$, we get that (1) is equivalent to (3), which concludes the proof. □

We conclude this section with a comment on the equality between $\underline{w}$ and $\overline{w}$ in the bounded framework.

Corollary 3.1.3. *Let the maps $\underline{w}$ and $\overline{w}$ be given by* (3.1.1) *and* (3.1.2)*, respectively. If $G\in\mathcal{C}_b^+(E)$, then we get $\underline{w}\equiv\overline{w}$.*

Proof. Let $x\in E$ and $\tau\in\mathcal{T}_x$ be such that $\mathbb{E}_x\left[e^{\int_0^\tau g(X_s)ds}\right] < \infty$. Recalling (3.1.9) and the non-negativity of g, for any $T\ge 0$, we get

$$Z(\tau\wedge T) \le e^{\int_0^\tau g(X_s)ds} e^{\|G\|}.$$

Thus, using Lebesgue's dominated convergence theorem, we get the uniform integrability of $(Z(\tau\wedge T))$, $T\ge 0$. Hence, from Lemma 3.1.2 and the quasi left-continuity of Z, we get

$$\liminf_{T\to\infty} \mathbb{E}_x\left[Z(\tau\wedge T)\right] = \mathbb{E}_x\left[\liminf_{T\to\infty} Z(\tau\wedge T)\right] = \mathbb{E}_x\left[Z_\tau\right]. \tag{3.1.14}$$

Also, note that, from the non-negativity of G, we get that the stopping time $\tau\in\mathcal{T}_x$ satisfying $\mathbb{E}_x\left[e^{\int_0^\tau g(X_s)ds}\right] = \infty$ cannot be optimal for $\overline{w}$ and $\underline{w}$. Thus, taking infimum over $\tau\in\mathcal{T}_x$ in (3.1.14) and using Proposition 3.1.1, we get $\underline{w}\equiv\overline{w}$, which concludes the proof. □

Remark 3.1.4. It should be noted that the statement of Corollary 3.1.3 may not hold if G is unbounded from above. The specific examples can be found in Section 5.3. $\diamondsuit$

3.2 Discrete time stopping

In this section, we consider discrete-time versions of the problems associated with (3.1.1) and (3.1.2). We introduce a suitable form of the Bellman equation and show that the value functions of the corresponding optimal stopping problems are its solutions. Also, we present certain results related to the finite time horizon and dyadic optimal stopping.

In this section, by $((X_n)_{n\in\mathbb{N}}, (\mathbb{P}_x)_{x\in E})$ we denote a standard discrete-time $\mathcal{C}_b$-Feller-Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. By analogy to (3.1.1) and (3.1.2), we define

$$\underline{w}(x) := \inf_{\tau\in\mathcal{T}_x^0} \ln \mathbb{E}_x \left[\exp\left(\sum_{i=0}^{\tau-1} g(X_i) + G(X_\tau)\right)\right], \quad x \in E, \tag{3.2.1}$$

$$\overline{w}(x) := \inf_{\tau\in\mathcal{T}_{x,b}^0} \ln \mathbb{E}_x \left[\exp\left(\sum_{i=0}^{\tau-1} g(X_i) + G(X_\tau)\right)\right], \quad x \in E. \tag{3.2.2}$$

We show specific regularity properties of the maps $\underline{w}$ and $\overline{w}$ and we link them to suitable finite time horizon stopping problems. Also, at the end of this section we return to the continuous time framework and comment on the dyadic optimal stopping setting.

In this section we assume ($\mathcal{A}1$) together with the following discrete time versions of the conditions introduced in Section 3.1:

($\mathcal{A}2'$) (Integrability). For any $n \in \mathbb{N}$ and $x \in E$, we have $\mathbb{E}_x\left[e^{G(X_n)}\right] < \infty$.

($\mathcal{A}3'$) (Continuity). For any $n \in \mathbb{N}$ and $h \in \mathcal{C}^+(E)$ satisfying $h(x) \le G(x)$, $x \in E$, it holds that the map $x \mapsto \mathbb{E}_x\left[e^{h(X_n)}\right]$ is continuous.

Assumptions ($\mathcal{A}2'$)–($\mathcal{A}3'$) could be seen as discrete time counterparts of ($\mathcal{A}2$)–($\mathcal{A}3$). Also, it should be noted that if G is bounded, then, recalling the $\mathcal{C}_b$-Feller property, we get that ($\mathcal{A}2'$) and ($\mathcal{A}3'$) are automatically satisfied. Finally, note that in this section we do not need any version of ($\mathcal{A}4$).

3.2.1 Discrete time Bellman equation

The results in this section are linked to properties of maps $w \in \mathcal{M}^+(E)$ satisfying the discrete time dynamic programming principle of the form

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}^0} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau \wedge n-1} g(X_i) + 1_{\{\tau < n\}} G(X_\tau) + 1_{\{\tau \geq n\}} w(X_n)} \right], \quad x \in E,\, n \in \mathbb{N}. \tag{3.2.3}$$

Typically, a solution to this equation could be linked to a value function of some infinite time horizon optimal stopping problem. More specifically, Equation (3.2.3) could be seen as an identity quantifying the fact that the associated optimisation problem could be split into two sub-periods: before the time $n \in \mathbb{N}$ and after this moment. If a decision-maker decides to stop the process before n, the pay-off consists of the cost for waiting (linked to g) and the terminal reward/cost measured by G. If the decision-maker decides not to stop the process before n, the pay-off consists of the cost for waiting and $w(X_n)$, which is the (averaged) optimal value of the problem restarted at X_n. A more extensive discussion on the idea of the dynamic programming principle in various optimisation contexts can be found e.g. in Hernandez-Lerma and Lasserre (1996).

Due to the fact that in this section we act in the discrete time framework, Equation (3.2.3) could be simplified to the discrete time Bellman equation of the form

$$e^{w(x)} = \min(e^{G(x)}, e^{g(x)} \mathbb{E}_x[e^{w(X_1)}]), \quad w \in \mathcal{M}^+(E),\, x \in E; \tag{3.2.4}$$

note that the equivalence of (3.2.3) and (3.2.4) is proved in Lemma 3.2.2. Also, let us note that (3.2.4) could be expressed in the operator form as

$$e^{w(x)} = Se^w(x), \quad w \in \mathcal{M}^+(E),\, x \in E, \tag{3.2.5}$$

where $S \colon \mathcal{M}^+(E) \to \mathcal{M}^+(E)$ is the Bellman operator given by

$$Sh(x) := \min(e^{G(x)}, e^{g(x)} \mathbb{E}_x[h(X_1)]), \quad h \in \mathcal{M}^+(E),\, x \in E. \tag{3.2.6}$$

The use of this operator simplifies some arguments related to the Bellman equation; see e.g. the proofs of Proposition 3.2.4 and Theorem 3.2.6.

Now, let us state a martingale characterisation of solutions to (3.2.4); see Lemma 3.2.1. The proof is relatively standard; yet, we include it for completeness.

Lemma 3.2.1. *Let $w \in \mathcal{M}^+(E)$ be a solution to (3.2.4) and let*

$$z_w(n) := \exp\left(\sum_{i=0}^{n-1} g(X_i) + w(X_n)\right), \quad n \in \mathbb{N}, \tag{3.2.7}$$

$$\tau_w := \inf\{n \in \mathbb{N} : w(X_n) = G(X_n)\}. \tag{3.2.8}$$

Then, for any $x \in E$, the process $(z_w(n))$, $n \in \mathbb{N}$, is a $\mathbb{P}_x$-submartingale and $(z_w(\tau_w \wedge n))$, $n \in \mathbb{N}$, is a $\mathbb{P}_x$-martingale.

Proof. First, note that directly from (3.2.4), we get $e^{g(x)}\mathbb{E}_x\left[e^{w(X_1)}\right] \geq e^{w(x)}$, $x \in E$. Thus, using the Markov property, for any $x \in E$ and $n \in \mathbb{N}$, we get

$$\begin{aligned}\mathbb{E}_x\left[z_w(n+1)|\mathcal{F}_n\right] &= e^{\sum_{i=0}^{n-1} g(X_i)}e^{g(X_n)}\mathbb{E}_{X_n}\left[e^{w(X_1)}\right]\\ &\geq e^{\sum_{i=0}^{n-1} g(X_i)}e^{w(X_n)} = z_w(n),\end{aligned}$$

which shows the submartingale property of $(z_w(n))$, $n \in \mathbb{N}$.

Second, note that from (3.2.4), on the event $\{\tau_w > n\}$, we get

$$e^{w(X_n)} = e^{g(X_n)}\mathbb{E}_{X_n}\left[e^{w(X_1)}\right].$$

Thus, for any $x \in E$ and $n \in \mathbb{N}$, we get

$$\begin{aligned}&\mathbb{E}_x[z_w(\tau_w \wedge (n+1))|\mathcal{F}_n]\\ &\qquad= 1_{\{\tau_w \leq n\}}z_w(\tau_w) + 1_{\{\tau_w > n\}}e^{\sum_{i=0}^{n} g(X_i)}\mathbb{E}_x\left[e^{w(X_{n+1})}|\mathcal{F}_n\right]\\ &\qquad= 1_{\{\tau_w \leq n\}}z_w(\tau_w) + 1_{\{\tau_w > n\}}e^{\sum_{i=0}^{n-1} g(X_i)}e^{g(X_n)}\mathbb{E}_{X_n}\left[e^{w(X_1)}\right]\\ &\qquad= 1_{\{\tau_w \leq n\}}z_w(\tau_w \wedge n) + 1_{\{\tau_w > n\}}e^{\sum_{i=0}^{\tau_w \wedge n-1} g(X_i)}e^{w(X_{\tau_w \wedge n})}\\ &\qquad= z_w(\tau_w \wedge n),\end{aligned}$$

which concludes the proof. □

In Lemma 3.2.2 we show that (3.2.3) and (3.2.4) are equivalent.

Lemma 3.2.2. *A map $w \in \mathcal{M}^+(E)$ is a solution to (3.2.3) if and only if it is a solution to (3.2.4).*

Proof. First, let $w \in \mathcal{M}^+(E)$ be a solution to (3.2.4). Also, let the process $(z_w(n))$, $n \in \mathbb{N}$, be given by (3.2.7). Then, using Lemma 3.2.1 and Doob's optional stopping theorem, we get that, for any stopping time $\tau \in \mathcal{T}^0$, the process $(z_w(\tau \wedge n))$, $n \in \mathbb{N}$, is a submartingale. Then, noting that, from (3.2.4), we get $w(\cdot) \le G(\cdot)$, for any $x \in E$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} e^{w(x)} &\le \inf_{\tau \in \mathcal{T}^0} \mathbb{E}_x[z_w(\tau \wedge n)] \\ &= \inf_{\tau \in \mathcal{T}^0} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau \wedge n-1} g(X_i) + w(X_{\tau \wedge n})} \right] \\ &\le \inf_{\tau \in \mathcal{T}^0} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau \wedge n-1} g(X_i) + 1_{\{\tau < n\}} G(X_{\tau_w}) + 1_{\{\tau \ge n\}} w(X_n)} \right]. \end{aligned} \tag{3.2.9}$$

Also, using again Lemma 3.2.1, we get that the process $(z_w(\tau_w \wedge n))$, $n \in \mathbb{N}$, is a martingale, where τ_w is given by (3.2.8). Thus, noting that, on the event $\{\tau_w < n\}$, we get $w(X_{\tau_w}) = G(X_{\tau_w})$, for any $x \in E$ and $n \in \mathbb{N}$, we also get

$$\begin{aligned} e^{w(x)} &= \mathbb{E}_x[z_w(\tau_w \wedge n)] \\ &= \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau_w \wedge n-1} g(X_i) + w(X_{\tau_w \wedge n})} \right] \\ &= \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau_w \wedge n-1} g(X_i) + 1_{\{\tau_w < n\}} G(X_\tau) + 1_{\{\tau_w \ge n\}} w(X_n)} \right]. \end{aligned}$$

Combining this with (3.2.9), for any $x \in E$ and $n \in \mathbb{N}$, we get

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}^0} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau \wedge n-1} g(X_i) + 1_{\{\tau < n\}} G(X_\tau) + 1_{\{\tau \ge n\}} w(X_n)} \right].$$

Thus, the map w satisfies (3.2.3).

Second, let $w \in \mathcal{M}^+(E)$ be a solution to (3.2.3). Then, setting $n = 1$ in (3.2.3), we get

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}^0} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau \wedge 1-1} g(X_i) + 1_{\{\tau < 1\}} G(X_\tau) + 1_{\{\tau \ge 1\}} w(X_n)} \right], \quad x \in E.$$

In particular, we get that we may restrict our attention to stopping times $\tau \in \mathcal{T}^0$ such that $\tau \in \{0, 1\}$. Thus, using Blumenthal's zero-one law, we get

$$e^{w(x)} = \min(e^{G(x)}, e^{g(x)} \mathbb{E}_x[e^{w(X_1)}]), \quad x \in E.$$

Hence, the map w is a solution to (3.2.4). □

Remark 3.2.3. Note that the proofs of Lemma 3.2.1 and Lemma 3.2.2 are valid even if we consider a generic $g \in \mathcal{C}_b(E)$. In particular, we do not need the condition $g(\cdot) \ge c > 0$. ◊

3.2.2 Solution to the problem

Now, we focus on the construction of solutions to (3.2.4). In particular, we show that the value functions from (3.2.1) and (3.2.2) are solutions to this equation. In this way we obtain some regularity properties of $\underline{w}$ and $\overline{w}$.

We start with finding the minimal and maximal solutions to (3.2.4). Recalling the non-negativity of the functions g and G, we get

$$0 \leq \underline{w}(x) \leq \overline{w}(x) \leq G(x), \quad x \in E.$$

Thus, to get the extremal solutions to (3.2.4), we iterate the operator S given by (3.2.6) on the lower and upper bounds of $\underline{w}$ and $\overline{w}$. More specifically, we recursively define the families of functions $(\underline{w}_n)_{n\in\mathbb{N}}$ and $(\overline{w}_n)_{n\in\mathbb{N}}$ by

$$\underline{w}_0(x) := 0, \qquad \underline{w}_{n+1}(x) := \ln Se^{\underline{w}_n}(x), \qquad n \in \mathbb{N},\, x \in E, \tag{3.2.10}$$

$$\overline{w}_0(x) := G(x), \qquad \overline{w}_{n+1}(x) := \ln Se^{\overline{w}_n}(x), \qquad n \in \mathbb{N},\, x \in E. \tag{3.2.11}$$

In Theorem 3.2.6 we show that the limits $\lim_{n\to\infty} \underline{w}_n$ and $\lim_{n\to\infty} \overline{w}_n$ are well defined and solve (3.2.5). Before we do this, in Proposition 3.2.4, we show that the sequences $(\underline{w}_n)$ and $(\overline{w}_n)$ could be linked to the value functions of finite time horizon optimal stopping problems. Note that in the proposition, $\inf_{\tau\leq n}$ denotes the infimum over the family of stopping times with values in $\{0, 1, \ldots, n\}$.

Proposition 3.2.4. *For any $n \in \mathbb{N}$, let the maps $\underline{w}_n$ and $\overline{w}_n$ be given by (3.2.10) and (3.2.11), respectively. Then:*

(1) *We get*

$$\underline{w}_n(x) = \inf_{\tau\leq n} \ln \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + 1_{\{\tau<n\}} G(X_\tau)} \right], \quad n \in \mathbb{N},\, x \in E. \tag{3.2.12}$$

Also, for any $x \in E$, the sequence $(\underline{w}_n(x))$ is increasing and, for any $n \in \mathbb{N}$, the map $x \mapsto \underline{w}_n(x)$ is continuous. Next, for any $n \in \mathbb{N}$ and $x \in E$, the stopping time

$$\underline{\tau}_n := \inf \left\{ k \in \mathbb{N}\colon \underline{w}_{n-k}(X_k) = G(X_k) \right\} \wedge n \tag{3.2.13}$$

is optimal for $\overline{w}_n(x)$. Moreover, for any $n \in \mathbb{N}$ and $x \in E$, the process

$$\underline{z}_n(k) := e^{\sum_{i=0}^{k-1} g(X_i) + \underline{w}_{n-k}(X_k)}, \quad k = 0, \ldots, n,$$

is a $\mathbb{P}_x$-submartingale and $(\underline{z}_n(\underline{\tau}_n \wedge k))$, $k = 0, \ldots, n$, is a $\mathbb{P}_x$-martingale.

(2) *We get*

$$\overline{w}_n(x) = \inf_{\tau\leq n} \ln \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i)+G(X_\tau)} \right], \quad n\in\mathbb{N},\, x\in E. \tag{3.2.14}$$

Also, for any $x \in E$, the sequence $(\overline{w}_n(x))$ is decreasing and, for any $n\in\mathbb{N}$, the map $x\mapsto \overline{w}_n(x)$ is continuous. Next, for any $n\in\mathbb{N}$ and $x\in E$, the stopping time

$$\overline{\tau}_n := \inf\{k\in\mathbb{N}\colon \overline{w}_{n-k}(X_k) = G(X_k)\} \tag{3.2.15}$$

is optimal for $\overline{w}_n(x)$. Moreover, for any $n\in\mathbb{N}$ and $x\in E$, the process

$$\overline{z}_n(k) := e^{\sum_{i=0}^{k-1} g(X_i)+\overline{w}_{n-k}(X_k)}, \quad k=0,\ldots,n,$$

is a $\mathbb{P}_x$-submartingale and $(\overline{z}_n(\overline{\tau}_n\wedge k))$, $k=0,\ldots,n$, is a $\mathbb{P}_x$-martingale.

Proof. First, we show the properties of $\underline{w}_n$. Identity (3.2.12), the optimality of $\underline{\tau}_n$ and the martingale characterisation follow from Proposition A.1.1 used with $h\equiv 0$. The monotonicity of $\underline{w}_n(x)$ follows from an induction argument. Indeed, since $g(\cdot)\geq 0$ and $G(\cdot)\geq 0$, we get

$$e^{\underline{w}_1(x)} = e^{g(x)}\mathbb{E}_x\left[e^{\underline{w}_0(X_1)}\right]\wedge e^{G(x)} \geq 1 = e^{\underline{w}_0(x)}, \quad x\in E.$$

Thus, assuming that, for some $n\in\mathbb{N}$, we get $\underline{w}_{n+1}(x)\geq \underline{w}_n(x)$, $x\in E$, and, using the monotonicity of S, we get

$$e^{\underline{w}_{n+2}(x)} = Se^{\underline{w}_{n+1}}(x) \geq Se^{\underline{w}_n}(x) = e^{\underline{w}_{n+1}(x)}, \quad x\in E.$$

Also, recursively using (3.2.10) and Assumption ($\mathcal{A}3'$), we get the continuity of $x\mapsto \underline{w}_n(x)$ for any $n\in\mathbb{N}$.

Second, we focus on $\overline{w}_n$. To get (3.2.14), the optimality of $\overline{\tau}_n$ and the martingale characterisation, it is enough to use again Proposition A.1.1 with $h\equiv G$. Also, from (3.2.11) and Assumption ($\mathcal{A}3'$), we get the continuity of $x\mapsto\overline{w}_n(x)$ for any $n\in\mathbb{N}$. Finally, the monotonicity of $n\mapsto \overline{w}_n(x)$, $x\in E$, follows from the fact that in (3.2.14), when we increase n, we also increase the family of stopping times that we minimise over. □

Remark 3.2.5. Note that in the proof of Proposition 3.2.4, Assumption ($\mathcal{A}3'$) was used only to show the continuity property. In fact, the remaining claims hold true even if we omit this assumption and the $\mathcal{C}_b$-Feller property of the process. ♢

Based on the monotonicity properties stated in Proposition 3.2.4, we get that the limits $\lim_{n\to\infty}\underline{w}_n(x)$ and $\lim_{n\to\infty}\overline{w}_n(x)$, $x\in E$, are well defined. In fact, in Theorem 3.2.6 we link these limits to the maps $\underline{w}$ and $\overline{w}$ and show that they satisfy the Bellman equation.

Theorem 3.2.6. *Let the maps $\underline{w}$ and $\overline{w}$ be given by (3.2.1) and (3.2.2), respectively. Also, for any $n\in\mathbb{N}$, let the maps $\underline{w}_n$ and $\overline{w}_n$ be given by (3.2.10) and (3.2.11), respectively. Then:*

(1) *For any $x\in E$, we get*

$$\underline{w}(x)=\lim_{n\to\infty}\underline{w}_n(x)\quad\textit{and}\quad \overline{w}(x)=\lim_{n\to\infty}\overline{w}_n(x).$$

Also, $\underline{w}$ is lower semi-continuous and $\overline{w}$ is upper semi-continuous.

(2) *The maps $\underline{w}$ and $\overline{w}$ are solutions to (3.2.4).*

(3) *For any solution $w\in\mathcal{M}^+(E)$ to the Bellman equation (3.2.4), we get $\underline{w}(\cdot)\le w(\cdot)\le\overline{w}(\cdot)$.*

Proof. For transparency, we split the proof into three steps: (1) proof of $\underline{w}(x)=\lim_{n\to\infty}\underline{w}_n(x)$, $x\in E$, the fact that $\underline{w}$ satisfies (3.2.4), and the lower semi-continuity of $\underline{w}$; (2) proof of $\overline{w}(x)=\lim_{n\to\infty}\overline{w}_n(x)$, $x\in E$, the fact that $\overline{w}$ satisfies (3.2.4), and the upper semi-continuity of $\overline{w}$; (3) proof that $\underline{w}$ and $\overline{w}$ are minimal and maximal solutions to the Bellman equation, respectively.

STEP 1. We show that $\underline{w}(x)=\lim_{n\to\infty}\underline{w}_n(x)$, $x\in E$, the map $\underline{w}$ satisfies (3.2.4), and is lower semi-continuous. Using the monotonicity property from Proposition 3.2.4, we get that the map $\underline{\widehat{w}}(x):=\lim_{n\to\infty}\underline{w}_n(x)$, $x\in E$, is well defined. Also, letting $n\to\infty$ in (3.2.10) and using (3.2.5), we get that $\underline{\widehat{w}}$ is a solution to (3.2.4). Next, using Proposition 3.2.4 and recalling the non-negativity of g and G, for any $n\in\mathbb{N}$ and $x\in E$, we get

$$\begin{aligned}e^{\underline{w}_n(x)}&=\inf_{\tau\in\mathcal{T}_x^0}\mathbb{E}_x\left[e^{\sum_{i=0}^{\tau\wedge n-1}g(X_i)+1_{\{\tau<n\}}G(X_\tau)}\right]\\&\le\inf_{\tau\in\mathcal{T}_x^0}\mathbb{E}_x\left[e^{\sum_{i=0}^{\tau-1}g(X_i)+G(X_\tau)}\right]=e^{\underline{w}(x)}.\end{aligned}$$

Thus, letting $n\to\infty$, we get $\underline{\widehat{w}}\le\underline{w}$. Now, let us define

$$\underline{z}(n):=\exp\left(\sum_{i=0}^{n-1}g(X_i)+\underline{\widehat{w}}(X_n)\right),\quad n\in\mathbb{N},\tag{3.2.16}$$

$$\underline{\tau}:=\inf\{n\in\mathbb{N}:\underline{\widehat{w}}(X_n)=G(X_n)\},\tag{3.2.17}$$

and note that, by Lemma 3.2.1, the process $(\underline{z}(\underline{\tau}\wedge n))$, $n\in\mathbb{N}$, is a martingale. Also, noting that $\widehat{\underline{w}}(\cdot)\geq 0$, recalling that $g(\cdot)\geq c>0$, and using Fatou's lemma, for any $x\in E$, we get

$$\begin{aligned}\mathbb{E}_x\left[e^{\underline{\tau}c}\right]=\mathbb{E}_x\left[\liminf_{n\to\infty}e^{(\underline{\tau}\wedge n)c}\right]&\leq\mathbb{E}_x\left[\liminf_{n\to\infty}e^{\sum_{i=0}^{\underline{\tau}\wedge n-1}g(X_i)+\widehat{\underline{w}}(X_{\underline{\tau}\wedge n})}\right]\\&\leq\liminf_{n\to\infty}\mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}\wedge n-1}g(X_i)+\widehat{\underline{w}}(X_{\underline{\tau}\wedge n})}\right]\\&=\mathbb{E}_x\left[\underline{z}(0)\right]=e^{\widehat{\underline{w}}(x)}\leq e^{G(x)}<\infty.\end{aligned}$$

In particular, we get $\mathbb{P}_x[\underline{\tau}<\infty]=1$ and $\underline{\tau}\in\mathcal{T}_x^0$ for any $x\in E$. Thus, noting that $\widehat{\underline{w}}(X_{\underline{\tau}})=G(X_{\underline{\tau}})$, we get

$$\begin{aligned}e^{\underline{w}(x)}&\leq\mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}-1}g(X_i)+G(X_{\underline{\tau}})}\right]\\&=\mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}-1}g(X_i)+\widehat{\underline{w}}(X_{\underline{\tau}})}\right]\\&=\mathbb{E}_x\left[\liminf_{n\to\infty}e^{\sum_{i=0}^{\underline{\tau}\wedge n-1}g(X_i)+\widehat{\underline{w}}(X_{\underline{\tau}}\wedge n)}\right]\\&\leq\liminf_{n\to\infty}\mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}\wedge n-1}g(X_i)+\widehat{\underline{w}}(X_{\underline{\tau}}\wedge n)}\right]=\mathbb{E}_x\left[\underline{z}(0)\right]=e^{\widehat{\underline{w}}(x)},\end{aligned}\tag{3.2.18}$$

hence $\underline{w}\equiv\widehat{\underline{w}}$. Also, recalling that $\widehat{\underline{w}}$ solves (3.2.4), we get that $\underline{w}$ is a solution to the Bellman equation. Finally, noting that, by Proposition 3.2.4, the map $\underline{w}$ is the increasing limit of the continuous functions, we get that $\underline{w}$ is lower semi-continuous, which concludes the proof of this step.

Step 2. We show that $\overline{w}(x)=\lim_{n\to\infty}\overline{w}_n(x)$, $x\in E$, the map $\overline{w}$ satisfies (3.2.4), and is upper semi-continuous. Using a discrete time version of Proposition 3.1.1, we get

$$\overline{w}(x)=\inf_{\tau\in\mathcal{T}_x^0}\liminf_{k\to\infty}\ln\mathbb{E}_x\left[e^{\sum_{i=0}^{\tau\wedge k-1}g(X_i)+G(X_{\tau\wedge k})}\right],\quad x\in E.\tag{3.2.19}$$

Thus, recalling Proposition 3.2.4, for any $n\in\mathbb{N}$ and $x\in E$, we get

$$\begin{aligned}e^{\overline{w}(x)}&\leq\inf_{\tau\leq n}\liminf_{k\to\infty}\mathbb{E}_x\left[e^{\sum_{i=0}^{\tau\wedge k-1}g(X_i)+G(X_{\tau\wedge k})}\right]\\&=\inf_{\tau\leq n}\mathbb{E}_x\left[e^{\sum_{i=0}^{\tau-1}g(X_i)+G(X_\tau)}\right]=e^{\overline{w}_n(x)}.\end{aligned}$$

Consequently, letting $n\to\infty$, we get $\overline{w}(x)\leq\lim_{n\to\infty}\overline{w}_n(x)$, $x\in E$. Also, for any $n\in\mathbb{N}$ and $\hat{\tau}\in\mathcal{T}_x^0$, we get

$$\inf_{\tau\leq n}\mathbb{E}_x\left[e^{\sum_{i=0}^{\tau-1}g(X_i)+G(X_\tau)}\right]\leq\mathbb{E}_x\left[e^{\sum_{i=0}^{\hat{\tau}\wedge n-1}g(X_i)+G(X_{\hat{\tau}\wedge n})}\right].$$

Thus, letting $n \to \infty$, taking infimum over $\widehat{\tau} \in \mathcal{T}_x^0$, and recalling (3.2.19), we get $\lim_{n\to\infty} \overline{w}_n(x) \leq \overline{w}(x)$, $x \in E$, and consequently $\lim_{n\to\infty} \overline{w}_n(x) = \overline{w}(x)$, $x \in E$. Also, letiing $n \to \infty$ in (3.2.11), we get that $\lim_{n\to\infty} \overline{w}_n$ solves (3.2.4). Thus, the map $\overline{w}$ is a solution to the Bellman equation and, as the decreasing limit of the continuous functions $x \mapsto \overline{w}_n(x)$, is upper semi-continuous, which concludes the proof of this step.

STEP 3. We show that $\underline{w}$ and $\overline{w}$ are minimal and maximal solution to the Bellman equation. Let $w \in \mathcal{M}^+(E)$ be a solution to (3.2.4). In particular, recalling the operator S given by (3.2.6), we get $e^{w(\cdot)} = Se^w(\cdot)$ and $0 \leq w(\cdot) \leq G(\cdot)$. Then, recursively applying the operator S and recalling (3.2.10) and (3.2.11), for any $n \in \mathbb{N}$, we get $\underline{w}_n(\cdot) \leq w(\cdot) \leq \overline{w}_n(\cdot)$. Thus, letting $n \to \infty$, we get $\underline{w}(\cdot) \leq w(\cdot) \leq \overline{w}(\cdot)$, which concludes the proof. $\square$

Remark 3.2.7. It should be noted that if we omit Assumption ($\mathcal{A}3'$), all claims of Theorem 3.2.6 remain valid, except for the semi-continuity properties of $\underline{w}$ and $\overline{w}$; see Remark 3.2.5 for a similar discussion. $\lozenge$

Remark 3.2.8. In Theorem 3.2.6 we showed that both $\underline{w}$ and $\overline{w}$ satisfy the Bellman equation. *A priori*, we might suspect that these maps are identical. However, in Example 5.3.1 we show that in general $\underline{w} \neq \overline{w}$. Sufficient conditions for the identity $\underline{w} \equiv \overline{w}$ are discussed in Theorem 3.2.11. $\lozenge$

Remark 3.2.9. From Theorem 3.2.6 we deduce that, in the unbounded case, the family of finite time horizon stopping problems may not converge to their infinite time horizon versions. More specifically, from Proposition 3.2.4, we get that the map

$$\overline{w}_n(x) = \inf_{\tau \leq n} \ln \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + G(X_\tau)} \right], \quad n \in \mathbb{N},\, x \in E,$$

may be seen as a finite time horizon counterpart of

$$\underline{w}(x) = \inf_{\tau \in \mathcal{T}_x^0} \ln \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + G(X_\tau)} \right], \quad x \in E,$$

with stopping times bounded by $n \in \mathbb{N}$. Thus, one might conjecture that $\overline{w}_n$ converges to $\underline{w}$ as $n \to \infty$. However, from Theorem 3.2.6, we get $\overline{w}_n \to \overline{w}$ as $n \to \infty$, where $\overline{w}$ is given by (3.2.2), and from Example 5.3.1 we see that in general $\underline{w} \neq \overline{w}$. Also, note that Theorem 3.2.6 provides a finite time horizon approximation scheme for $\underline{w}$; this can be done with the help of the family $\underline{w}_n$ from (3.2.10). $\lozenge$

From the proof of Theorem 3.2.6, we get a useful corollary about an optimal stopping time for $\underline{w}$.

Corollary 3.2.10. *Let the map $\underline{w}$ be given by (3.2.1). Then, for any $x \in E$, the stopping time*

$$\underline{\tau} = \inf\{n \in \mathbb{N} : \underline{w}(X_n) = G(X_n)\} \in \mathcal{T}_x^0 \tag{3.2.20}$$

is optimal for $\underline{w}(x)$.

Proof. This follows directly from (3.2.18); note that there we also showed that $\underline{\tau} \in \mathcal{T}_x^0$, $x \in E$. □

Now we formulate a sufficient condition for the identity $\underline{w} = \overline{w}$. Recalling Theorem 3.2.6, we get that in this case the Bellman equation (3.2.4) admits a unique solution. To simplify the notation, we define the process

$$Z(n) := \exp\left(\sum_{i=0}^{n-1} g(X_i) + G(X_n)\right), \quad n \in \mathbb{N}. \tag{3.2.21}$$

Theorem 3.2.11. *Let the maps $\underline{w}$ and $\overline{w}$ be given by (3.2.1) and (3.2.2), respectively. Also, let $\underline{\tau}$ and $(Z(n))$, $n \in \mathbb{N}$, be given by (3.2.20) and (3.2.21), respectively. If, for any $x \in E$, the process $(Z(\underline{\tau} \wedge n))$, $n \in \mathbb{N}$, is $\mathbb{P}_x$-uniformly integrable, then we get $\underline{w} \equiv \overline{w}$.*

Proof. Recall that, by Corollary 3.2.10, the stopping time $\underline{\tau}$ is optimal for $\underline{w}(x)$, $x \in E$. In particular, we get $\underline{\tau} \in \mathcal{T}_x^0$, $x \in E$. Then, using a discrete time version of Proposition 3.1.1 and the uniform integrability of $(Z(\underline{\tau} \wedge n))$, $n \in \mathbb{N}$, for any $x \in E$, we get

$$e^{\overline{w}(x)} \leq \lim_{n\to\infty} \mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}\wedge n-1} g(X_i)+G(X_{\underline{\tau}\wedge n})}\right] = \mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}-1} g(X_i)+G(X_{\underline{\tau}})}\right] = e^{\underline{w}(x)}.$$

Recalling that we always get $\underline{w} \leq \overline{w}$, we conclude the proof. □

Remark 3.2.12. Let $\overline{\tau} := \inf\{t \geq 0 : \overline{w}(X_t) = G(X_t)\}$. Based on the condition from Theorem 3.2.11, one may ask if the uniform integrability of $(Z(\overline{\tau} \wedge n))$ is also sufficient for $\underline{w} \equiv \overline{w}$. However, as discussed in Remark 5.3.3, this is not the case. ◊

As we show in Proposition 3.2.13, the uniform integrability condition from Theorem 3.2.11 is satisfied e.g. for a bounded terminal cost function G.

Proposition 3.2.13. *Let $G \in \mathcal{C}_b^+(E)$. Then, for any $x \in E$, the process $(Z(\underline{\tau} \wedge n))$, $n \in \mathbb{N}$, from Theorem 3.2.11, is $\mathbb{P}_x$-uniformly integrable.*

Proof. Recalling Corollary 3.2.10, we get that the stopping time $\underline{\tau}$ is optimal for $\underline{w}$. In particular, recalling the non-negativity of G, for any $x \in E$, we get

$$\mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}-1} g(X_i)}\right] \leq \mathbb{E}_x\left[e^{\sum_{i=0}^{\underline{\tau}-1} g(X_i)+G(X_{\underline{\tau}})}\right] = e^{\underline{w}(x)} \leq e^{G(x)} < \infty.$$

Then, recalling the non-negativity of g, noting that $Z(n\wedge\underline{\tau}) \leq e^{\sum_{i=0}^{\underline{\tau}-1} g(X_i)}e^{\|G\|}$, $n \in \mathbb{N}$, and using Lebesgue's dominated convergence theorem, we get the $\mathbb{P}_x$-uniform integrability of $(Z(n \wedge \underline{\tau}))$ for any $x \in E$, which concludes the proof. □

For ease of reference, in Theorem 3.2.14 we summarise the properties of the optimal stopping problems with a bounded terminal cost function G. In this case, we get a unique solution to the Bellman equation, which can be used to prove the continuity of the value functions $\underline{w}$ and $\overline{w}$.

Theorem 3.2.14. *Let $G \in \mathcal{C}_b^+(E)$ and let the maps $\underline{w}$ and $\overline{w}$ be given by (3.2.1) and (3.2.2), respectively. Also, let $w \in \mathcal{M}^+(E)$ be a solution to the Bellman equation (3.2.4). Then, we get*

$$\underline{w} \equiv w \equiv \overline{w} \in \mathcal{C}_b^+(E).$$

Also, for any $x \in E$, the stopping time

$$\underline{\tau} = \inf\{n \in \mathbb{N} : \underline{w}(X_n) = G(X_n)\} \in \mathcal{T}_x^0$$

is optimal for $\underline{w}$. Moreover, for any $x \in E$, the process

$$z_{\underline{w}}(n) := \exp\left(\sum_{i=0}^{n-1} g(X_i) + \underline{w}(X_n)\right), \quad n \in \mathbb{N}$$

is a $\mathbb{P}_x$-submartingale and $(z_{\underline{w}}(\underline{\tau} \wedge n))$, $n \in \mathbb{N}$, is a $\mathbb{P}_x$-martingale.

Proof. Using Proposition 3.2.13 and Theorem 3.2.11, we get $\underline{w} \equiv \overline{w}$. This, combined with Theorem 3.2.6, shows the uniqueness of a solution to the Bellman equation (3.2.4). Also, using the semi-continuity properties from Theorem 3.2.6, we get that $\underline{w} \equiv \overline{w} \in \mathcal{C}_b^+(E)$. Finally, recalling Corollary 3.2.10 and Lemma 3.2.1, we conclude the proof. □

3.2.3 Dyadic optimal stopping

Now, we show some results related to the dyadic optimal stopping problem. In this case we consider a continuous time process, but the stopping times are restricted to a discrete time-grid. This provides a link between discrete and continuous time settings. Throughout this section $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ denotes a continuous time standard $\mathcal{C}_b$-Feller–Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Also, recall the time step $\delta_m := \frac{1}{2^m}$ and the respective families of stopping times $\mathcal{T}_x^m$ and $\mathcal{T}_{x,b}^m$, $m \in \mathbb{N}$, $x \in E$, from Section 2.1.

By analogy to (3.2.6) and (3.2.5), for any $m \in \mathbb{N}$, we define the dyadic Bellman operator and the Bellman equation

$$S^m h(x) := \min(e^{G(x)}, \mathbb{E}_x[e^{\int_0^{\delta_m} g(X_s)ds} h(X_{\delta_m})]), \quad h \in \mathcal{M}^+(E),\, x \in E, \tag{3.2.22}$$

$$e^{w(x)} = S^m e^w(x), \quad w \in \mathcal{M}^+(E),\, x \in E, \tag{3.2.23}$$

respectively. The properties of a solution to the Bellman equation (3.2.23) are summarised in Theorem 3.2.15. In the theorem, for simplicity, we restrict our attention to a bounded terminal cost function G and we assume only ($\mathcal{A}1$). The unbounded case, under some regularity assumptions, could be treated using similar logic.

Theorem 3.2.15. *Let $G \in \mathcal{C}_b^+(E)$, $m \in \mathbb{N}$, and let $w^m \in \mathcal{M}^+(E)$ be a solution to (3.2.23). Then, we get*

$$\begin{aligned} w^m(x) &= \inf_{\tau \in \mathcal{T}_x^m} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau) \right) \right] \\ &= \inf_{\tau \in \mathcal{T}_{x,b}^m} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau) \right) \right], \quad x \in E. \end{aligned}$$

Also, for any $x \in E$, the stopping time

$$\tau^m := \delta_m \cdot \inf\{n \in \mathbb{N} : w^m(X_{n\delta_m}) = G(X_{n\delta_m})\} \in \mathcal{T}^m$$

is optimal for $w^m(x)$.

Proof. The proof follows the lines of the argument leading to Theorem 3.2.14, thus we provide only an outline. Fix some $m \in \mathbb{N}$. By analogy to (3.2.10) and (3.2.11), we define

$$\underline{w}_0^m(x) := 0, \qquad \underline{w}_{n+1}^m(x) := \ln S^m e^{\underline{w}_n^m}(x), \qquad n \in \mathbb{N},\, x \in E, \tag{3.2.24}$$

$$\overline{w}_0^m(x) := G(x), \qquad \overline{w}_{n+1}^m(x) := \ln S^m e^{\overline{w}_n^m}(x), \qquad n \in \mathbb{N},\, x \in E. \tag{3.2.25}$$

As in Proposition 3.2.4, we may show that

$$\underline{w}_n^m(x) = \inf_{\substack{\tau \leq n\delta_m \\ \tau \in \mathcal{T}_x^m}} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + 1_{\{\tau < n\delta_m\}} G(X_\tau)} \right], \quad n \in \mathbb{N},\, x \in E,$$

$$\overline{w}_n^m(x) = \inf_{\substack{\tau \leq n\delta_m \\ \tau \in \mathcal{T}_x^m}} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)} \right], \quad n \in \mathbb{N},\, x \in E,$$

and the maps $x \mapsto \underline{w}_n^m(x)$ and $x \mapsto \overline{w}_n^m(x)$ are continuous; note that the continuity property follows from (3.2.24)–(3.2.25), the $\mathcal{C}_b$-Feller property, and Proposition 2.1.8.

Next, as in Theorem 3.2.6, we may show that $\lim_{n\to\infty} \underline{w}_n^m(x) = \underline{w}^m(x)$, $x \in E$, and $\lim_{n\to\infty} \overline{w}_n^m(x) = \overline{w}^m(x)$, $x \in E$, where

$$\underline{w}^m(x) := \inf_{\tau \in \mathcal{T}_x^m} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)} \right], \quad n \in \mathbb{N},\, x \in E,$$

$$\overline{w}^m(x) := \inf_{\tau \in \mathcal{T}_{x,b}^m} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)} \right], \quad n \in \mathbb{N},\, x \in E.$$

Also, the maps $x \mapsto \underline{w}^m(x)$ and $x \mapsto \overline{w}^m(x)$ are lower semi-continuous and upper semi-continuous, respectively. Moreover, letting $n \to \infty$ in (3.2.24)–(3.2.25), we get that $\underline{w}^m$ and $\overline{w}^m$ are solutions to (3.2.23). Furthermore, as in Theorem 3.2.11 and Proposition 3.2.13, we show that from the boundedness of G, we get $\underline{w}^m \equiv \overline{w}^m$. Thus, there is a unique solution to the dyadic Bellman equation (3.2.23) and this solution is continuous. The optimality of τ^m could be shown by the argument used in Corollary 3.2.10. □

3.3 Finite time horizon continuous time stopping

In this section, we consider finite time horizon continuous time optimal stopping problems. We show the continuity of the corresponding value functions with respect to a time horizon and a starting point of the process. Next, we give a characterisation of optimal stopping times. Also, at the end of this section we show how to extend our results to the case where the cost functions depend on time.

3.3.1 Time-homogeneous case

In this section, $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a continuous time standard $\mathcal{C}_b$-Feller-Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Our main focus is set on the finite time horizon continuous time optimal stopping problems. More specifically, by analogy to (3.2.12) and (3.2.14), for any $T \geq 0$, we define

$$\underline{w}_T(x) := \inf_{\tau \leq T} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + 1_{\{\tau < T\}} G(X_\tau) \right) \right], \quad x \in E, \quad (3.3.1)$$

$$\overline{w}_T(x) := \inf_{\tau \leq T} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau) \right) \right], \quad x \in E. \quad (3.3.2)$$

where $\inf_{\tau \leq T}$ denotes the infimum over stopping times with values in $[0, T]$.

In this section we assume ($\mathcal{A}1$)–($\mathcal{A}4$). However, it should be noted that here we do not need the condition $g(\cdot) \geq c > 0$ from ($\mathcal{A}1$). Also, for (3.3.1), we may consider a generic $g \in \mathcal{C}_b(E)$, i.e. to study properties of the map $(T, x) \mapsto \underline{w}_T(x)$ we do not need the non-negativity condition of g; see Remark 3.3.3 for details.

We start with a useful convergence result for the terminal cost function. In a nutshell, it says that, for a bounded G, a finite time horizon $T \geq 0$, a stopping time $\tau \leq T$, and a family of stopping times (τ_n) decreasing uniformly to τ, we get $\mathbb{E}_x \left| e^{G(X_{\tau_n})} - e^{G(X_\tau)} \right| \to 0$ as $n \to \infty$, and the convergence is uniform with respect to x from some compact set and the choice of τ and (τ_n). In the proof of this result we extensively use the distance control properties from Assumption ($\mathcal{A}4$).

Lemma 3.3.1. *Let $G \in \mathcal{C}_b^+(E)$. Let (a_n) be a sequence of non-negative numbers such that $a_n \downarrow 0$ as $n \to \infty$. For any bounded stopping time $\tau \in \mathcal{T}$, let us define $\mathcal{T}(\tau, a_n) := \{\sigma \in \mathcal{T} : 0 \leq \sigma - \tau \leq a_n\}$, $n \in \mathbb{N}$. Then, for any $T \geq 0$ and a compact set $\Gamma \subset E$, we get*

$$\lim_{n\to\infty} \sup_{x \in \Gamma} \sup_{\substack{\tau \leq T \\ \sigma \in \mathcal{T}(\tau, a_n)}} \mathbb{E}_x \left| e^{G(X_\sigma)} - e^{G(X_\tau)} \right| = 0.$$

Proof. Let $T \geq 0$, $\Gamma \subset E$ be a compact set, and $\varepsilon > 0$. Using Assumption ($\mathcal{A}4$), we get that there exists $R > 0$ such that

$$\sup_{x \in \Gamma} \mathbb{P}_x \left[\sup_{t \in [0,T]} \rho(X_t, x) \geq R \right] \leq \varepsilon. \quad (3.3.3)$$

For brevity, we set $Z(t,s) := \left|e^{G(X_t)} - e^{G(X_s)}\right|$, $t, s \geq 0$. By (3.3.3), we get

$$
\begin{aligned}
&\sup_{x\in\Gamma} \sup_{\substack{\tau\leq T \\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x\left[Z(\sigma,\tau)\right] \\
&\quad\leq \sup_{x\in\Gamma} \sup_{\substack{\tau\leq T \\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x\left[1_{\{\rho(X_\tau,x)\geq R\}} Z(\sigma,\tau) + 1_{\{\rho(X_\tau,x)<R\}} Z(\sigma,\tau)\right] \\
&\qquad\leq 2\varepsilon e^{\|G\|} + \sup_{x\in\Gamma} \sup_{\substack{\tau\leq T \\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x\left[1_{\{\rho(X_\tau,x)<R\}} Z(\sigma,\tau)\right].
\end{aligned}
\tag{3.3.4}
$$

Set $B := \{x \in E \colon \rho(\Gamma, x) \leq R+1\}$. Since $e^{G(\cdot)}$ is uniformly continuous on B, we can find $r > 0$ such that

$$
\sup_{y,z\in B\colon \rho(y,z)\leq r} \left|e^{G(y)} - e^{G(z)}\right| \leq \varepsilon. \tag{3.3.5}
$$

Recalling that $a_n \downarrow 0$ and using Assumption ($\mathcal{A}4$), we may find $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, we get

$$
\sup_{x\in B} \mathbb{P}_x\left[\sup_{t\in[0,a_n]} \rho(X_t, x) \geq r\right] \leq \varepsilon. \tag{3.3.6}
$$

Also, using the strong Markov property and the fact that for $\sigma \in \mathcal{T}(\tau, a_n)$ we get $0 \leq \sigma - \tau \leq a_n$, we get

$$
\begin{aligned}
&\sup_{x\in\Gamma} \sup_{\substack{\tau\leq T \\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x\left[1_{\{\rho(X_\tau,x)<R\}} Z(\sigma,\tau)\right] \\
&\qquad\qquad\leq \sup_{x\in\Gamma} \sup_{\substack{\tau\leq T \\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x\left[1_{\{\rho(X_\tau,x)<R\}} \mathbb{E}_{X_\tau}\left[\sup_{t\in[0,a_n]} Z(t,0)\right]\right].
\end{aligned}
$$

Thus, recalling (3.3.5) and (3.3.6), for $n \geq n_0$, we get

$$
\begin{aligned}
&\mathbb{E}_x\left[1_{\{\rho(X_\tau,x)<R\}} \mathbb{E}_{X_\tau}\left[1_{\{\sup_{t\in[0,a_n]}\rho(X_t,X_0)<r\}} \sup_{t\in[0,a_n]} Z(t,0)\right]\right] \leq \varepsilon, \\
&\mathbb{E}_x\left[1_{\{\rho(X_\tau,x)<R\}} \mathbb{E}_{X_\tau}\left[1_{\{\sup_{t\in[0,a_n]}\rho(X_t,X_0)\geq r\}} \sup_{t\in[0,a_n]} Z(t,0)\right]\right] \leq 2\varepsilon e^{\|G\|},
\end{aligned}
$$

where the upper bounds are uniform with respect to $x \in \Gamma$, $\tau \leq T$, and $\sigma \in \mathcal{T}(\tau, a_n)$. Consequently, for $n \geq n_0$, we get

$$
\sup_{x\in\Gamma} \sup_{\substack{\tau\leq T \\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x\left[1_{\{\rho(X_\tau,x)<R\}} Z(\sigma,\tau)\right] \leq \varepsilon + 2\varepsilon e^{\|G\|}. \tag{3.3.7}
$$

Combining (3.3.4) with (3.3.7), for $n \geq n_0$, we get

$$\sup_{x\in\Gamma} \sup_{\substack{\tau\leq T\\ \sigma\in\mathcal{T}(\tau,a_n)}} \mathbb{E}_x \left| e^{G(X_\sigma)} - e^{G(X_\tau)} \right| \leq \varepsilon(4e^{\|G\|}+1), \tag{3.3.8}$$

which concludes the proof. □

In Theorem 3.3.2 we show the properties of the maps $(T,x) \mapsto \underline{w}_T(x)$ and $(T,x) \mapsto \overline{w}_T(x)$. This may be seen as a continuous time version of Proposition 3.2.4.

Theorem 3.3.2. *For any $T \geq 0$, let the maps $\underline{w}_T$ and $\overline{w}_T$ be given by (3.3.1) and (3.3.2), respectively. Then:*

(1) *The map $(T,x) \mapsto \underline{w}_T(x)$ is jointly continuous and, for any $x \in E$, the map $T \mapsto \underline{w}_T(x)$ is increasing. Also, for any $T \geq 0$ and $x \in E$, the stopping time*

$$\underline{\tau}_T := \inf\left\{t \geq 0 : \underline{w}_{T-t}(X_t) = G(X_t)\right\} \wedge T \tag{3.3.9}$$

is optimal for $\underline{w}_T(x)$. Moreover, for any $T \geq 0$ and $x \in E$, the process

$$\underline{z}_T(t) := e^{\int_0^{t\wedge T} g(X_s)ds + \underline{w}_{T-t\wedge T}(X_{t\wedge T})}, \quad t \geq 0, \tag{3.3.10}$$

is a $\mathbb{P}_x$-submartingale and $(\underline{z}_T(\underline{\tau}_T \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale.

(2) *The map $(T,x) \mapsto \overline{w}_T(x)$ is jointly continuous and, for any $x \in E$, the map $T \mapsto \overline{w}_T(x)$ is decreasing. Also, for any $T \geq 0$ and $x \in E$, the stopping time*

$$\overline{\tau}_T := \inf\left\{t \geq 0 : \overline{w}_{T-t}(X_t) = G(X_t)\right\} \tag{3.3.11}$$

is optimal for $\overline{w}_T(x)$. Moreover, for any $T \geq 0$ and $x \in E$, the process

$$\overline{z}_T(t) := e^{\int_0^{t\wedge T} g(X_s)ds + \overline{w}_{T-t\wedge T}(X_{t\wedge T})}, \quad t \geq 0,$$

is a $\mathbb{P}_x$-submartingale and $(\overline{z}_T(\overline{\tau}_T \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale.

Proof. The proof is relatively complex, thus we start with some general comments and the outline of the structure of the argument.

First, we show the monotonicity properties of $T \mapsto \underline{w}_T(x)$ and $T \mapsto \overline{w}_T(x)$ with a fixed $x \in E$. To see that the map $T \mapsto \overline{w}_T(x)$ is decreasing, it is enough to note that, when we increase T, we also increase the family of stopping times that we minimise over. Also, note that for this property we do not need the non-negativity of g. To see that $T \mapsto \underline{w}_T(x)$ is increasing, let $T \geq 0$, $u \in [0, T]$, and let $\tau_\varepsilon \leq T$ be an ε-optimal stopping time for $\underline{w}_T(x)$. Then, using the non-negativity of g and G, we get

$$\begin{aligned}\underline{w}_{T-u}(x) &\leq \ln \mathbb{E}_x\left[e^{\int_0^{\tau_\varepsilon \wedge (T-u)} g(X_s)ds + 1_{\{\tau_\varepsilon < T-u\}} G(X_{\tau_\varepsilon})}\right] \\ &\leq \ln \mathbb{E}_x\left[e^{\int_0^{\tau_\varepsilon} g(X_s)ds + 1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon})}\right] \leq \underline{w}_T(x) + \varepsilon. \end{aligned} \tag{3.3.12}$$

Letting $\varepsilon \to 0$, we conclude that $T \mapsto \underline{w}_T(x)$ is increasing.

Second, note that if we show the joint continuity of $(T, x) \mapsto \underline{w}_T(x)$ and $(T, x) \mapsto \overline{w}_T(x)$, using Proposition A.2.8, we get the optimality of $\underline{\tau}_T$ and $\overline{\tau}_T$, and the martingale properties of the processes $\underline{z}_T$ and $\overline{z}_T$. Also, using Lemma A.2.9, to prove the joint continuity of $(T, x) \mapsto \underline{w}_T(x)$, it is enough to show the continuity of $T \mapsto \underline{w}_T(x)$ with a fixed $x \in E$ and the continuity of $x \mapsto \underline{w}_T(x)$ with a fixed $T \geq 0$; a similar property holds for $(T, x) \mapsto \underline{w}_T(x)$.

The rest of the argument consists of two major parts. First, we show the statement under the additional assumption that G is bounded. Then, using a suitable approximation, we relax the boundedness condition. More specifically, for any $n \in \mathbb{N}$, let us define bounded versions of (3.3.1) and (3.3.2) by

$$\begin{aligned}\underline{w}_T^n(x) &:= \inf_{\tau \leq T} \ln \mathbb{E}_x\left[e^{\int_0^\tau g(X_s)ds + 1_{\{\tau < T\}} G(X_\tau) \wedge n}\right], \quad T \geq 0,\, x \in E, \\ \overline{w}_T^n(x) &:= \inf_{\tau \leq T} \ln \mathbb{E}_x\left[e^{\int_0^\tau g(X_s)ds + G(X_\tau) \wedge n}\right], \quad T \geq 0,\, x \in E.\end{aligned}$$

In the following we show that the maps $T \mapsto \underline{w}_T^n(x)$, $T \mapsto \overline{w}_T^n(x)$, $x \mapsto \underline{w}_T^n(x)$, and $x \mapsto \overline{w}_T^n(x)$ are continuous. Also, we show that

$$\underline{w}_T(x) = \lim_{n\to\infty} \underline{w}_T^n(x) \text{ and } \overline{w}_T(x) = \lim_{n\to\infty} \overline{w}_T^n(x),\, T \geq 0,\, x \in E, \tag{3.3.13}$$

and use these to show the properties of $(T, x) \mapsto \underline{w}_T(x)$ and $(T, x) \mapsto \overline{w}_T(x)$. Note that we already showed that, for a fixed $x \in E$ and $n \in \mathbb{N}$, the map $T \mapsto \underline{w}_T^n(x)$ is increasing and the map $T \mapsto \overline{w}_T^n(x)$ is decreasing; see the discussion at the beginning of this proof.

We split the remaining part of the proof into four steps: (1) proof of the continuity of $T \mapsto \underline{w}_T^n(x)$ and $T \mapsto \overline{w}_T^n(x)$ with a fixed $n \in \mathbb{N}$ and $x \in E$; (2) proof of the continuity of $x \mapsto \underline{w}_T^n(x)$ and $x \mapsto \overline{w}_T^n(x)$ with a fixed $n \in \mathbb{N}$ and $T \geq 0$; (3) proof of (3.3.13); (4) proof of the joint continuity of $(T, x) \mapsto \underline{w}_T(x)$ and $(T, x) \mapsto \overline{w}_T(x)$. At each step, we focus on the properties of $\underline{w}_T$ and its bounded version $\underline{w}_T^n$. The arguments for $\overline{w}_T$ and $\overline{w}_T^n$ are similar, thus we provide only an outline of the necessary modifications.

Step 1. We show that, for any $n \in \mathbb{N}$ and $x \in E$, the maps $T \mapsto \underline{w}^n_T(x)$ and $T \mapsto \overline{w}^n_T(x)$ are continuous. We start with the properties of $T \mapsto \underline{w}^n_T(x)$. Let us fix $n \in \mathbb{N}$ and $x \in E$. First, we show the left-continuity of $T \mapsto \underline{w}^n_T(x)$. Let $\varepsilon > 0$. For any $u \in [0,T]$, let $\tau^u_\varepsilon \leq T-u$ be an ε-optimal stopping time for $e^{\underline{w}^n_{T-u}(x)}$ and let

$$\underline{Z}^n_T(t) := e^{\int_0^{t\wedge T} g(X_s)ds + 1_{\{t<T\}}G(X_t)\wedge n}, \quad t \geq 0.$$

Since $T \mapsto \underline{w}^n_T(x)$ is increasing and $\underline{w}^n_T(x) \leq \ln \mathbb{E}_x\left[\underline{Z}^n_T(\tau^u_\varepsilon + u)\right]$, we get

$$\begin{aligned} 0 \leq e^{\underline{w}^n_T(x)} - e^{\underline{w}^n_{T-u}(x)} &\leq \mathbb{E}_x\left|\underline{Z}^n_T(\tau^u_\varepsilon + u) - \underline{Z}^n_{T-u}(\tau^u_\varepsilon)\right| + \varepsilon \\ &\leq \sup_{\tau \leq T} \mathbb{E}_x\left|\underline{Z}^n_T(\tau + u) - \underline{Z}^n_{T-u}(\tau)\right| + \varepsilon. \end{aligned} \tag{3.3.14}$$

Now, let us show that, for any $T > 0$ and $x \in E$, we get

$$\sup_{\tau \leq T} E_x\left|\underline{Z}^n_T(\tau + u) - \underline{Z}^n_{T-u}(\tau)\right| \to 0, \quad u \downarrow 0. \tag{3.3.15}$$

For any $x \in E$ and $\tau \leq T$, we get

$$\begin{aligned} &\mathbb{E}_x\left|\underline{Z}^n_T(\tau + u) - \underline{Z}^n_{T-u}(\tau)\right| \\ &\quad = \mathbb{E}_x\left|e^{\int_0^{(\tau+u)\wedge T} g(X_s)ds}\left(e^{1_{\{\tau+u<T\}}G(X_{\tau+u})\wedge n} - e^{1_{\{\tau<T-u\}}G(X_\tau)\wedge n}\right)\right. \\ &\qquad\quad \left. + e^{1_{\{\tau<T-u\}}G(X_\tau)\wedge n + \int_0^{\tau\wedge(T-u)} g(X_s)ds}\left(e^{\int_{\tau\wedge(T-u)}^{(\tau+u)\wedge T} g(X_s)ds} - 1\right)\right| \\ &\qquad\quad \leq e^{T\|g\|}\,\mathbb{E}_x\left|e^{G(X_{\tau+u})\wedge n} - e^{G(X_\tau)\wedge n}\right| + e^{n+T\|g\|}\left(e^{u\|g\|} - 1\right). \end{aligned}$$

Noting that $G(\cdot) \wedge n \in \mathcal{C}^+_b(E)$ and using Lemma 3.3.1, we get

$$\sup_{\tau \leq T} \mathbb{E}_x\left|e^{G(X_{\tau+u})\wedge n} - e^{G(X_\tau)\wedge n}\right| \to 0, \quad u \downarrow 0.$$

Consequently, since $e^{u\|g\|} - 1 \to 0$ as $u \downarrow 0$, we get (3.3.15). Thus, recalling (3.3.14) and noting that ϵ was arbitrary, we get the left-continuity of $T \mapsto \underline{w}^n_T(x)$.

Second, we show the right-continuity of $T \mapsto \underline{w}_T(x)$. As in the first part of the proof, let $\tau_\epsilon \leq T$ be an ε-optimal stopping time for $\underline{w}^n_T(x)$. Using the monotonicity of $T \mapsto \underline{w}^n_T(x)$ and the boundedness of g and $G(\cdot) \wedge n$, we get

$$\begin{aligned} \underline{w}^n_T(x) &\leq \lim_{u\downarrow 0} \underline{w}^n_{T+u}(x) \\ &\leq \lim_{u\downarrow 0} \ln \mathbb{E}_x\left[e^{\int_0^{\tau_\varepsilon+u} g(X_s)ds + 1_{\{\tau_\varepsilon+u<T+u\}}G(X_{\tau_\varepsilon+u})\wedge n}\right] \\ &= \ln \mathbb{E}_x\left[e^{\int_0^{\tau_\varepsilon} g(X_s)ds + 1_{\{\tau_\varepsilon<T\}}G(X_{\tau_\varepsilon})\wedge n}\right] \leq \underline{w}^n_T(x) + \varepsilon; \end{aligned} \tag{3.3.16}$$

note that, in the third line, we used Lebesgue's dominated convergence theorem and the fact that X is right-continuous. Letting $\varepsilon \to 0$, we get the right-continuity of $T \to \underline{w}^n_T(x)$.

Now, we focus on the continuity of the map $T \mapsto \overline{w}^n_T(x)$. The proof is similar to the argument for $T \mapsto \underline{w}^n_T(x)$, and we provide only an outline. For the left-continuity, let $\tau_\epsilon \leq T$ be an ε-optimal stopping time for $\overline{w}^n_T(x)$. Using the monotonicity of $T \mapsto \overline{w}^n_T(x)$ and the boundedness of g and $G(\cdot) \wedge n$, we get

$$\begin{aligned}
\overline{w}^n_T(x) &\leq \lim_{u \downarrow 0} \overline{w}^n_{T-u}(x) \\
&\leq \lim_{u \downarrow 0} \ln \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon \wedge (T-u)} g(X_s)ds + G(X_{\tau_\varepsilon \wedge (T-u)}) \wedge n} \right] \\
&= \ln \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon} g(X_s)ds + G(X_{\tau_\varepsilon}) \wedge n} \right] \leq \overline{w}^n_T(x) + \varepsilon; \qquad (3.3.17)
\end{aligned}$$

where in the last line we used Lebesgue's dominated convergence theorem and the fact that X is quasi left-continuous. Letting $\varepsilon \to 0$, we get that $T \mapsto \overline{w}^n_T(x)$ is left-continuous.

For the right-continuity, set $\overline{Z}^n_T(t) := e^{\int_0^{t \wedge T} g(X_s)ds + G(X_{t \wedge T}) \wedge n}$, $t \geq 0$. Also, let $u, \varepsilon > 0$ and τ^ε be an ε-optimal stopping time for $e^{\overline{w}^n_{T+u}(x)}$. Then, as in (3.3.14), we get

$$\begin{aligned}
0 \leq e^{\overline{w}^n_T(x)} - e^{\overline{w}^n_{T+u}(x)} &\leq \mathbb{E}_x \left| \overline{Z}^n_T(\tau^\varepsilon) - \overline{Z}^n_{T+u}(\tau^\varepsilon) \right| + \varepsilon \\
&\leq \sup_{\tau \leq T+u} \mathbb{E}_x \left| \overline{Z}^n_T(\tau) - \overline{Z}^n_{T+u}(\tau) \right| + \varepsilon. \qquad (3.3.18)
\end{aligned}$$

Also, note that, for any $x \in E$, $u \leq 1$, $x \in E$, and $\tau \leq T + u$, we get

$$\begin{aligned}
\mathbb{E}_x \left| \overline{Z}^n_T(\tau) - \overline{Z}^n_{T+u}(\tau) \right| = \mathbb{E}_x \Big| & e^{\int_0^{\tau \wedge (T+u)} g(X_s)ds} \left(e^{G(X_{\tau \wedge T}) \wedge n} - e^{G(X_{\tau \wedge (T+u)}) \wedge n} \right) \\
&+ e^{G(X_\tau \wedge T) \wedge n + \int_0^{\tau \wedge T} g(X_s)ds} \left(1 - e^{\int_{\tau \wedge T}^{\tau \wedge (T+u)} g(X_s)ds} \right) \Big| \\
\leq & e^{(T+1)\|g\|} \mathbb{E}_x \left| e^{G(X_{\tau \wedge T}) \wedge n} - e^{G(X_{\tau \wedge (T+u)}) \wedge n} \right| \\
&+ e^{n + T\|g\|} \left(e^{u\|g\|} - 1 \right). \qquad (3.3.19)
\end{aligned}$$

Thus, noting that $e^{u\|g\|} - 1 \to 0$ as $u \downarrow 0$ and using Lemma 3.3.1, as in (3.3.15), we get

$$\sup_{\tau \leq T+u} \mathbb{E}_x \left| \overline{Z}^n_T(\tau) - \overline{Z}^n_{T+u}(\tau) \right| \to 0, \quad u \downarrow 0.$$

Recalling (3.3.18) and letting $\varepsilon \to 0$, we get the right-continuity of $T \mapsto \overline{w}_T(x)$, which concludes the proof of this step.

Step 2. We show that, for any $n \in \mathbb{N}$ and $T \geq 0$, the maps $x \mapsto \underline{w}_T^n(x)$ and $x \mapsto \overline{w}_T^n(x)$ are continuous. We start with the properties of $x \mapsto \underline{w}_T^n(x)$. Let us fix $n \in \mathbb{N}$ and $T \geq 0$. We use a dyadic approximation of $\underline{w}_T$. For any $m \in \mathbb{N}$, we set

$$\underline{w}_T^{n,m}(x) := \inf_{\tau \in \mathcal{T}_T^m} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + 1_{\{\tau < T\}} G(X_\tau) \wedge n} \right], \quad x \in E, \tag{3.3.20}$$

where $\mathcal{T}_T^m$ is the family of stopping times taking values in $\{0, \frac{T}{2^m}, \frac{2T}{2^m}, \ldots, T\}$. We show that, for any $m \in \mathbb{N}$, the map $x \mapsto \underline{w}_T^{n,m}(x)$ is continuous. Let us fix $m \in \mathbb{N}$ and recursively define the sequence of functions

$$\begin{aligned}
&\widetilde{w}_T^{n,0}(x) := 0, \\
&e^{\widetilde{w}_T^{n,j}(x)} := \mathbb{E}_x \left[e^{\int_0^{\frac{T}{2^m}} g(X_s)ds + \widetilde{w}_T^{j-1}(X_{\frac{T}{2^m}})} \right] \wedge e^{G(x) \wedge n}, \quad j = 1, \ldots, 2^m.
\end{aligned}$$

Using Assumption ($\mathcal{A}3$) inductively, we get that, for any $j = 1, \ldots, 2^m$, the map $x \mapsto \widetilde{w}_T^{n,j}(x)$ is continuous. Also, using a dyadic version of Proposition A.1.1, we get $\underline{w}_T^{n,m} \equiv \widetilde{w}_T^{n,2^m}$, which implies the continuity of $x \mapsto \underline{w}_T^{n,m}(x)$.

We now show that $\lim_{m\to\infty} \underline{w}_T^{n,m}(x) = \underline{w}_T^n(x)$ uniformly in x from a compact set. This, together with the continuity of $x \mapsto \underline{w}_T^{n,m}(x)$, shows the continuity of $x \mapsto \underline{w}_T^n(x)$. Let $\varepsilon > 0$ and $\tau_\varepsilon \leq T$ be an ε-optimal stopping time for $e^{\underline{w}_T^n(x)}$. For any $m \in \mathbb{N}$, we define its $\mathcal{T}_T^m$ approximation by

$$\tau_\varepsilon^m := \inf\{\tau \in \mathcal{T}_T^m \colon \tau \geq \tau_\varepsilon\} = \textstyle\sum_{j=1}^{2^m} 1_{\left\{\frac{T}{2^m}(j-1) < \tau_\varepsilon \leq \frac{T}{2^m} j\right\}} \frac{T}{2^m} j.$$

Noting that $\tau_\varepsilon^m \leq T$, for any $m \in \mathbb{N}$ and $x \in E$, we get

$$\begin{aligned}
0 &\leq e^{\underline{w}_T^{n,m}(x)} - e^{\underline{w}_T^n(x)} \\
&\leq \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds + 1_{\{\tau_\varepsilon^m < T\}} G(X_{\tau_\varepsilon^m}) \wedge n} \right] \\
&\quad - \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon} g(X_s)ds + 1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon}) \wedge n} \right] + \varepsilon \\
&= \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds} \left(e^{1_{\{\tau_\varepsilon^m < T\}} G(X_{\tau_\varepsilon^m}) \wedge n} - e^{1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon}) \wedge n} \right) \right] \\
&\quad + \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon} g(X_s)ds + 1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon}) \wedge n} \left(e^{\int_{\tau_\varepsilon}^{\tau_\varepsilon^m} g(X_s)ds} - 1 \right) \right] + \varepsilon \\
&\leq \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds} \left(e^{1_{\{\tau_\varepsilon^m < T\}} G(X_{\tau_\varepsilon^m}) \wedge n} - e^{1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon}) \wedge n} \right) \right] \\
&\quad + \left(e^{\underline{w}_T^n(x)} + \varepsilon \right) \left(e^{\frac{T}{2^m} \|g\|} - 1 \right) + \varepsilon.
\end{aligned} \tag{3.3.21}$$

For any $T \geq 0$ and $x \in E$, we get

$$\left(e^{\underline{w}_T^n(x)} + \varepsilon\right)\left(e^{\frac{T}{2^m}\|g\|} - 1\right) \leq (e^n + \varepsilon)\left(e^{\frac{T}{2^m}\|g\|} - 1\right) \to 0, \quad m \to \infty; \tag{3.3.22}$$

in particular, this convergence is uniform in x. Also, noting that $\tau_\varepsilon^m \geq \tau_\varepsilon$ and using Lemma 3.3.1, we get

$$\begin{aligned}
&\mathbb{E}_x\left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds}\left(e^{1_{\{\tau_\varepsilon^m < T\}}G(X_{\tau_\varepsilon^m})\wedge n} - e^{1_{\{\tau_\varepsilon < T\}}G(X_{\tau_\varepsilon})\wedge n}\right)\right] \\
&\quad \leq \mathbb{E}_x\left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds}\left(e^{1_{\{\tau_\varepsilon < T\}}G(X_{\tau_\varepsilon^m})\wedge n} - e^{1_{\{\tau_\varepsilon < T\}}G(X_{\tau_\varepsilon})\wedge n}\right)\right] \\
&\quad \leq e^{T\|g\|}\,\mathbb{E}_x\left|e^{G(X_{\tau_\varepsilon^m})\wedge n} - e^{G(X_{\tau_\varepsilon})\wedge n}\right| \to 0, \quad m \to \infty;
\end{aligned} \tag{3.3.23}$$

also, the convergence is uniform in x from compact sets. Consequently, recalling that $\epsilon > 0$ was arbitrary, we conclude the proof of the continuity of $x \mapsto \underline{w}_T^n(x)$.

The argument for $x \mapsto \overline{w}_T^n(x)$, is similar to the one used for $x \mapsto \underline{w}_T(x)$, thus we provide only an outline. By analogy to (3.3.20), we define

$$\overline{w}_T^{n,m}(x) := \inf_{\tau \in \mathcal{T}_T^m} \mathbb{E}_x\left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)\wedge n}\right], \quad n, m \in \mathbb{N}.$$

Using the argument applied to $\underline{w}_T^{n,m}$, we show that the map $x \mapsto \overline{w}_T^{n,m}(x)$ is continuous. Also, as in (3.3.21)–(3.3.23), we show that

$$\begin{aligned}
0 &\leq e^{\overline{w}_T^{n,m}(x)} - e^{\overline{w}_T^n(x)} \\
&\leq e^{T\|g\|}\,\mathbb{E}_x\left|e^{G(X_{\tau_\varepsilon^m})\wedge n} - e^{G(X_{\tau_\varepsilon})\wedge n}\right| + (e^{\frac{T}{2^m}\|g\|} - 1)(e^n + \varepsilon) + \varepsilon,
\end{aligned}$$

where τ_ε is an ϵ-optimal stopping time for $e^{\overline{w}_T^n(x)}$ and τ_ε^m is the $\mathcal{T}_T^m$-dyadic approximation of τ_ε^m. Thus, using Lemma 3.3.1, we get that $\overline{w}_T^{n,m}(x)$ converges to $\overline{w}_T^n(x)$ as $m \to \infty$ uniformly in x from compact sets. This shows the continuity of $x \mapsto \overline{w}_T^n(x)$ and concludes the proof of this step.

Step 3. We show that $\underline{w}_T(x) = \lim_{n\to\infty} \underline{w}_T^n(x)$ and $\overline{w}_T(x) = \lim_{n\to\infty} \overline{w}_T^n(x)$, $T \geq 0$, $x \in E$. For brevity, we focus on $\underline{w}_T$ and $\underline{w}_T^n$; the argument for $\overline{w}_T$ and $\overline{w}_T^n$ follows the same lines. The main difficulty in this step stems from the fact that the sequence $(\underline{w}_T^n(x))_{n\in\mathbb{N}}$ is increasing. Thus, we need to interchange the supremum with respect to n with the infimum with respect to τ.

Let us fix $T \geq 0$ and $x \in E$. Also, let us define the family of events $A_n := \{\sup_{t\in[0,T]} G(X_t) \leq n\}$, $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we get $A_n \subset A_{n+1}$.

Moreover, using the càdlàg property of X, the continuity of G, and the fact that $T < \infty$, we get $\mathbb{P}_x[\cup_{n=1}^{\infty} A_n] = 1$, $x \in E$.

Combining Steps 1 and 2 with Lemma A.2.9, we get that, for any $n \in \mathbb{N}$, the map $(T, x) \mapsto \underline{w}_T^n(x)$ is jointly continuous. Thus, for any $n \in \mathbb{N}$, using Proposition A.2.8, we get that the stopping time

$$\underline{\tau}_T^n := \inf\{t \geq 0 : \underline{w}_{T-t}^n(X_t) = G(X_t) \wedge n\} \wedge T \tag{3.3.24}$$

is optimal for $\underline{w}_T^n(x)$. Also, note that, using the right-continuity of X, on the event $\{\underline{\tau}_T^n < T\}$, we get $\underline{w}_{T-\underline{\tau}_T^n}^n(X_{\underline{\tau}_T^n}) = G(X_{\underline{\tau}_T^n}) \wedge n$. Consequently, on the event $A_n \cap \{\underline{\tau}_T^n < T\}$, we get

$$\underline{w}_{T-\underline{\tau}_T^n}^{n+1}(X_{\underline{\tau}_T^n}) \geq \underline{w}_{T-\underline{\tau}_T^n}^n(X_{\underline{\tau}_T^n}) = G(X_{\underline{\tau}_T^n}) \wedge n = G(X_{\underline{\tau}_T^n}) \geq G(X_{\underline{\tau}_T^n}) \wedge (n+1).$$

Hence, noting that $\underline{w}_S^{n+1}(\cdot) \leq G(\cdot) \wedge (n+1)$, $S \geq 0$, we get $\underline{\tau}_T^{n+1} \leq \underline{\tau}_T^n$ on $A_n \cap \{\underline{\tau}_T^n < T\}$. In fact, we get $\underline{\tau}_T^{n+1} \leq \underline{\tau}_T^n$ on A_n; this follows from the fact that on $A_n \cap \{\underline{\tau}_T^n = T\}$, directly from (3.3.24), we get $\underline{\tau}_T^{n+1} \leq T = \underline{\tau}_T^n$. Acting inductively, for any $k \in \mathbb{N}$, we get $\underline{\tau}_T^{n+k+1} \leq \underline{\tau}_T^{n+k}$ on A_n. Thus, recalling that $\mathbb{P}_x[\cup_{n=1}^{\infty} A_n] = 1$, $x \in E$, we get that the limit $\widehat{\underline{\tau}}_T := \lim_{n\to\infty} \underline{\tau}_T^n$ is well defined. Also, using the right-continuity of X, we get

$$\lim_{n\to\infty} 1_{\{\underline{\tau}_T^n < T\}} G(X_{\underline{\tau}_T^n}) \wedge n = 1_{\{\widehat{\underline{\tau}}_T < T\}} G(X_{\widehat{\underline{\tau}}_T}).$$

Thus, using Fatou's lemma, we get

$$\begin{aligned}
e^{\underline{w}_T(x)} &\leq \mathbb{E}_x\left[e^{\int_0^{\widehat{\underline{\tau}}_T} g(X_s)ds + 1_{\{\widehat{\underline{\tau}}_T < T\}} G(X_{\widehat{\underline{\tau}}_T})}\right] \\
&= \mathbb{E}_x\left[\lim_{n\to\infty} e^{\int_0^{\underline{\tau}_T^n} g(X_s)ds + 1_{\{\underline{\tau}_T^n < T\}} G(X_{\underline{\tau}_T^n}) \wedge n}\right] \\
&\leq \lim_{n\to\infty} \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}_T^n} g(X_s)ds + 1_{\{\underline{\tau}_T^n < T\}} G(X_{\underline{\tau}_T^n}) \wedge n}\right] \\
&= \lim_{n\to\infty} e^{\underline{w}_T^n(x)} \leq e^{\underline{w}_T(x)},
\end{aligned} \tag{3.3.25}$$

which concludes the proof of $\underline{w}_T(x) = \lim_{n\to\infty} \underline{w}_T^n(x)$, $T \geq 0$, $x \in E$. The argument for $\overline{w}_T(x) = \lim_{n\to\infty} \overline{w}_T^n(x)$, $T \geq 0$, $x \in E$, follows the same logic and is omitted for brevity.

Step 4. We show the joint continuity of $(T, x) \mapsto \underline{w}_T(x)$ and $(T, x) \mapsto \overline{w}_T(x)$. We start with the properties of $(T, x) \mapsto \underline{w}_T(x)$. Recalling Lemma A.2.9, it is enough to show that the maps $T \mapsto \underline{w}_T(x)$ and $x \mapsto \underline{w}_T(x)$ are continuous.

First, we fix $x \in E$ and show the continuity of $T \mapsto \underline{w}_T(x)$. Recalling that by Steps 1 and 3, the function $T \mapsto \underline{w}_T(x)$ is the increasing limit of the continuous functions $T \mapsto \underline{w}_T^n(x)$, we get that $T \mapsto \underline{w}_T(x)$ is lower semi-continuous. This, together with the fact that $T \mapsto \underline{w}_T(x)$ is increasing, shows the left-continuity of $T \mapsto \underline{w}_T(x)$. Also, using the argument leading to (3.3.16), we get that $T \mapsto \underline{w}_T(x)$ is right-continuous; note that here, instead of the boundedness of $G(\cdot) \wedge n$, we use Assumption ($\mathcal{A}2$).

Second, we fix $T \geq 0$ and show the continuity of $x \mapsto \underline{w}_T(x)$. Again, recalling Steps 2 and 3, and the fact that the function $x \mapsto \underline{w}_T(x)$ is the increasing limit of the continuous functions $x \mapsto \underline{w}_T^n(x)$, we get that the function $x \mapsto \underline{w}_T(x)$ is lower semi-continuous. To show the upper semi-continuity we use a dyadic approximation of $\underline{w}_T$. For any $m \in \mathbb{N}$, we set

$$\widehat{\underline{w}}_T^m(x) := \inf_{\tau \in \mathcal{T}_T^m} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + 1_{\{\tau < T\}} G(X_\tau)} \right], \quad x \in E, \tag{3.3.26}$$

where $\mathcal{T}_T^m$ is the family of stopping times taking values in $\{0, \frac{T}{2^m}, \frac{2T}{2^m}, \ldots, T\}$. Using Assumption ($\mathcal{A}3$), as in Step 2, we show that $x \mapsto \widehat{\underline{w}}_T^m(x)$ is continuous. We now show that $\lim_{m\to\infty} \widehat{\underline{w}}_T^m(x) = \underline{w}_T(x)$ for any $x \in E$. This, together with the continuity of $x \mapsto \widehat{\underline{w}}_T^m(x)$ and the fact that $(\widehat{\underline{w}}_T^m(x))_{m\in\mathbb{N}}$ is decreasing, shows the upper semi-continuity of $x \mapsto \underline{w}_T(x)$. Let $\varepsilon > 0$ and $\tau_\varepsilon \leq T$ be an ε-optimal stopping time for $e^{\underline{w}_T(x)}$. For any $m \in \mathbb{N}$, we set $\tau_\varepsilon^m := \inf\{\tau \in \mathcal{T}_T^m \colon \tau \geq \tau_\varepsilon\} = \sum_{j=1}^{2^m} 1_{\left\{\frac{T}{2^m}(j-1) < \tau_\varepsilon \leq \frac{T}{2^m} j\right\}} \frac{T}{2^m} j$. As in (3.3.21), for any $x \in E$, we get

$$\begin{aligned}
0 &\leq e^{\widehat{\underline{w}}_T^m(x)} - e^{\underline{w}_T(x)} \\
&\leq \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds} \left(e^{1_{\{\tau_\varepsilon^m < T\}} G(X_{\tau_\varepsilon^m})} - e^{1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon})} \right) \right] \\
&\quad + \left(e^{\underline{w}_T(x)} + \varepsilon \right) \left(e^{\frac{T}{2^m} \|g\|} - 1 \right) + \varepsilon.
\end{aligned} \tag{3.3.27}$$

For any $x \in E$, we get $\left(e^{\underline{w}_T(x)} + \varepsilon\right) \left(e^{\frac{T}{2^m}\|g\|} - 1 \right) \to 0$ as $m \to \infty$. Also, noting that $\tau_\varepsilon^m \downarrow \tau_\varepsilon$ and using ($\mathcal{A}2$), for any $x \in E$, we get

$$\begin{aligned}
&\mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds} \left(e^{1_{\{\tau_\varepsilon^m < T\}} G(X_{\tau_\varepsilon^m})} - e^{1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon})} \right) \right] \\
&\quad \leq \mathbb{E}_x \left[e^{\int_0^{\tau_\varepsilon^m} g(X_s)ds} \left(e^{1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon^m})} - e^{1_{\{\tau_\varepsilon < T\}} G(X_{\tau_\varepsilon})} \right) \right] \\
&\quad \leq e^{T\|g\|} \, \mathbb{E}_x \left| e^{G(X_{\tau_\varepsilon^m})} - e^{G(X_{\tau_\varepsilon})} \right| \to 0, \quad m \to \infty;
\end{aligned} \tag{3.3.28}$$

note that here, in contrast to (3.3.23), the convergence is only pointwise. Consequently, letting $\epsilon \to 0$ in (3.3.27), we get that $x \mapsto \underline{w}_T(x)$ is continuous.

Let us now show the continuity of the map $(T, x) \mapsto \overline{w}_T(x)$. Again, recalling Lemma A.2.9, it is enough to show that the maps $T \mapsto \overline{w}_T(x)$ and $x \mapsto \overline{w}_T(x)$ are continuous. Combining Steps 1 and 3, we get that $T \mapsto \overline{w}_T(x)$, as the increasing limit of the continuous functions $T \mapsto \overline{w}^n_T(x)$, is lower semi-continuous. This, combined with the fact that $T \mapsto \overline{w}_T(x)$ is decreasing, shows the right-continuity of $T \mapsto \overline{w}_T(x)$. Using Assumption ($\mathcal{A}2$) and the argument leading to (3.3.17) we also get the left-continuity of $T \mapsto \overline{w}_T(x)$.

As in the case of $\underline{w}_T$, the lower semi-continuity of $x \mapsto \overline{w}_T(x)$ follows from the continuity of $x \mapsto \overline{w}^n_T(x)$ and the fact that $\overline{w}^n_T(x) \uparrow \overline{w}_T(x)$ as $n \to \infty$. For the upper semi-continuity, we consider a dyadic approximation of $\underline{w}_T$ given by

$$\widehat{\overline{w}}^m_T(x) := \inf_{\tau \in \mathcal{T}^m_T} \ln \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)} \right], \quad T \geq 0,\, m \in \mathbb{N},\, x \in E. \tag{3.3.29}$$

As in (3.3.27)–(3.3.28), we show that, for any $\varepsilon > 0$, we get

$$\begin{aligned} 0 &\leq e^{\widehat{\overline{w}}^m_T(x)} - e^{\overline{w}_T(x)} \\ &\leq \left(e^{\overline{w}_T(x)} + \varepsilon\right)\left(e^{\frac{T}{2^m}\|g\|} - 1\right) + e^{T\|g\|}\, \mathbb{E}_x \left| e^{G(X_{\tau^m_\varepsilon})} - e^{G(X_{\tau_\varepsilon})} \right| + \varepsilon, \end{aligned}$$

where τ_ε is an ε-optimal stopping time for $e^{\overline{w}_T(x)}$ and τ^m_ε is its $\mathcal{T}^m_T$ approximation. Thus, letting $m \to \infty$, we get a pointwise convergence of $\widehat{\overline{w}}^m_T(x)$ to $\overline{w}_T(x)$, which concludes the proof of the upper semi-continuity of $x \mapsto \overline{w}_T(x)$. □

Remark 3.3.3. It should be noted that the results for the map $\overline{w}_T$ remain valid even if we replace $g \in \mathcal{C}^+_b(E)$ by a generic $f \in \mathcal{C}_b(E)$. Indeed, for this map, the non-negativity of g was used only to show the convergence of the type of

$$\mathbb{E}_x |e^{\int_\tau^{\tau+h} g(X_s)ds} - 1| \leq e^{h\|g\|} - 1 \to 0, \quad h \downarrow 0;$$

see e.g. (3.3.19). For a generic $f \in \mathcal{C}_b(E)$ we may use the inequality

$$|e^y - e^z| \leq e^{\max(y,z)} |y - z|, \quad y, z \in \mathbb{R},$$

to get

$$\mathbb{E}_x |e^{\int_\tau^{\tau+h} f(X_s)ds} - 1| \leq e^{h\|f\|} h\|f\| \to 0, \quad h \downarrow 0.$$

However, similar argument cannot be applied for $\underline{w}_T$. Indeed, without the non-negativity condition for g it is difficult to obtain the monotonicity of the map $T \mapsto \underline{w}_T(x)$; see (3.3.12) for details. ◊

3.3.2 Time-inhomogeneous case

Now, we present some results related to the optimal stopping problems with time-dependent cost functions. Using a suitable state space extension technique, we show how to embed this type of problems into the setting considered in Theorem 3.3.2. More precisely, let $\widetilde{g}, \widetilde{G} \in \mathcal{C}_b^+([0,\infty) \times E)$. For any $T, t \geq 0$ and $x \in E$, we define

$$\underline{w}_T(t,x) := \inf_{\tau \leq T} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau \widetilde{g}(t+s, X_s) ds + 1_{\{\tau < T\}} \widetilde{G}(t+\tau, X_\tau) \right) \right], \tag{3.3.30}$$

$$\overline{w}_T(t,x) := \inf_{\tau \leq T} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau \widetilde{g}(t+s, X_s) ds + \widetilde{G}(t+\tau, X_\tau) \right) \right]. \tag{3.3.31}$$

Note that here, for brevity, we assumed that $\widetilde{G}$ is bounded. Under suitable integrability and continuity conditions, we could extend our results to the unbounded case; see the argument in Theorem 3.3.2.

The maps $(T,t,x) \mapsto \underline{w}_T(t,x)$ and $(T,t,x) \mapsto \overline{w}_T(t,x)$ could be seen as time-dependent versions of (3.3.1) and (3.3.2). In particular, using this formulation we may consider discounted risk-sensitive optimal stopping problems, where, for some discount factor $r > 0$, we set $\widetilde{g}(t,x) = e^{-rt} g(x)$ and $\widetilde{G}(t,x) = e^{-rt} G(x)$, $t \geq 0$, $x \in E$. Also, the properties of (3.3.30) are used to solve finite time horizon impulse control problems; see Section 4.2 for details.

The properties of the maps $(T,t,x) \mapsto \underline{w}_T(t,x)$ and $(T,t,x) \mapsto \overline{w}_T(t,x)$ are summarised in Theorem 3.3.4. This could be seen as a version of Theorem 3.3.2.

Theorem 3.3.4. *For any $T \geq 0$, let the maps $\underline{w}_T$ and $\overline{w}_T$ be given by (3.3.30) and (3.3.31), respectively. Then:*

(1) *The map $(T,t,x) \mapsto \underline{w}_T(t,x)$ is jointly continuous and bounded. Also, for any $T, t \geq 0$ and $x \in E$, the stopping time*

$$\underline{\tau}_T(t) := \inf \left\{ s \geq 0 : \underline{w}_{T-s}(t+s, X_s) = \widetilde{G}(t+s, X_s) \right\} \wedge T$$

is optimal for $\underline{w}_T(t,x)$. Moreover, for any $T, t \geq 0$ and $x \in E$, the process

$$\underline{z}_{T,t}(s) := e^{\int_0^{s \wedge T} \widetilde{g}(t+h, X_h) dh + \underline{w}_{T-s\wedge T}(t+s\wedge T, X_{s \wedge T})}, \quad s \geq 0,$$

is a $\mathbb{P}_x$-submartingale and $(\underline{z}_{T,t}(\underline{\tau}_T(t) \wedge s))$, $s \geq 0$, is a $\mathbb{P}_x$-martingale.

(2) *The map* $(T,t,x) \mapsto \overline{w}_T(t,x)$ *is jointly continuous and bounded. Also, for any* $T,t \geq 0$ *and* $x \in E$*, the stopping time*

$$\overline{\tau}_T(t) := \inf\left\{s \geq 0 : \overline{w}_{T-s}(t+s,X_s) = \widetilde{G}(t+s,X_s)\right\}$$

is optimal for $\overline{w}_T(t,x)$*. Moreover, for any* $T,t \geq 0$ *and* $x \in E$*, the process*

$$\overline{z}_{T,t}(s) := e^{\int_0^{s\wedge T}\widetilde{g}(t+h,X_h)dh+\overline{w}_{T-s\wedge T}(t+s\wedge T,X_{s\wedge T})}, \quad s \geq 0,$$

is a $\mathbb{P}_x$*-submartingale and* $(\overline{z}_{T,t}(\overline{\tau}_T(t) \wedge s))$*,* $s \geq 0$*, is a* $\mathbb{P}_x$*-martingale.*

Proof. We apply a space enlargement technique to use the results from Theorem 3.3.2. Let $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}})$ be a filtered measurable space, where $\widetilde{\Omega} := [0,\infty)\times\Omega$, the σ-field is given by $\widetilde{\mathcal{F}} := \mathcal{B}[0,\infty)\otimes\mathcal{F}$, and the filtration $\widetilde{\mathbb{F}} := (\widetilde{\mathcal{F}}_t)_{t\in[0,\infty)}$ is defined as $\widetilde{\mathcal{F}}_t := \mathcal{B}[0,\infty)\otimes\mathcal{F}_t$, $t \geq 0$. Also, let $\widetilde{E} := [0,\infty)\times E$, $\widetilde{\mathcal{E}} := \mathcal{B}[0,\infty)\otimes\mathcal{E}$, and $\widetilde{\rho}((t,x),(t',x')) := |t-t'| + \rho(x,x')$, $(t,x),(t',x') \in \widetilde{E}$. Next, for any $s \in [0,\infty)$ and $x \in E$, let us define $\widetilde{\mathbb{P}}_{(s,x)} := \delta_s \otimes \mathbb{P}_x$, where δ_s denotes the Dirac measure at s. Let us define the space-time process $\widetilde{X} = (\widetilde{X}_t)_{t\in[0,\infty)}$ by

$$\widetilde{X}_t(s,\omega) := (t+s, X_t(\omega)), \quad t \geq 0,\ (s,\omega) \in [0,\infty)\times\Omega = \widetilde{\Omega}. \tag{3.3.32}$$

Using the classic argument one can show that $((\widetilde{X}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_{(s,x)})_{(s,x)\in\widetilde{E}})$ is a standard Markov process on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}})$ with values in $(\widetilde{E},\widetilde{\mathcal{E}})$; see e.g. Section 1.4.6 in Shiryaev (1978) or Exercise 1.10 in Chapter III of Revuz and Yor (1999) for details. For transparency, we show that this process is $\mathcal{C}_b$-Feller and the version of Assumption ($\mathcal{A}4$) is satisfied for $\widetilde{X}$. More specifically, we define

$$\begin{aligned}
\widetilde{\mathcal{P}}_t h(y) &:= \widetilde{\mathbb{E}}_y[h(X_t)], \quad y \in \widetilde{E},\ h \in \mathcal{M}_b(\widetilde{E}),\\
\widetilde{M_{\widetilde{\Gamma}}}(t,r) &:= \sup_{y\in\widetilde{\Gamma}} \widetilde{\mathbb{P}}_y[\sup_{s\in[0,t]} \rho(\widetilde{X}_s,y) \geq r], \quad \widetilde{\Gamma} \subset \widetilde{E},\ t,r > 0.
\end{aligned}$$

and show that $\widetilde{\mathcal{P}}_t\mathcal{C}_b(\widetilde{E}) \subset \mathcal{C}_b(\widetilde{E})$, $t > 0$, and for any $t_0 > 0$, $r_0 > 0$, and a compact set $\widetilde{\Gamma} \subset \widetilde{E}$, we get $\lim_{t\to 0}\widetilde{M_{\widetilde{\Gamma}}}(t,r_0) = 0$ and $\lim_{r\to\infty}\widetilde{M_{\widetilde{\Gamma}}}(t_0,r) = 0$.

To show the $\mathcal{C}_b$-Feller property of $\widetilde{\mathcal{P}}_t$, let us fix $t > 0$ and $h \in \mathcal{C}_b(\widetilde{E})$. Also, let (s_n) and (x_n) be sequences such that $s_n \to s \in [0,\infty)$ and $x_n \to x \in E$ as $n \to \infty$. Using the $\mathcal{C}_b$-Feller property of $\mathcal{P}_t$, we get

$$|\widetilde{\mathcal{P}}_t h(s,x_n) - \widetilde{\mathcal{P}}_t h(s,x)| = |\mathbb{E}_{x_n}[h(t+s,X_t)] - \mathbb{E}_x[h(t+s,X_t)]| \to 0,\ n \to \infty.$$

Thus, using the inequality

$$|\widetilde{\mathcal{P}}_t h(s_n, x_n) - \widetilde{\mathcal{P}}_t h(s,x)| \leq |\widetilde{\mathcal{P}}_t h(s_n, x_n) - \widetilde{\mathcal{P}}_t h(s, x_n)| \\ + |\widetilde{\mathcal{P}}_t h(s, x_n) - \widetilde{\mathcal{P}}_t h(s,x)|, \tag{3.3.33}$$

it is enough to show $\sup_{z\in\Gamma} |\widetilde{\mathcal{P}}_t h(s_n, z) - \widetilde{\mathcal{P}}_t h(s,z)| \to 0$ as $n \to \infty$, where $\Gamma \subset E$ is some compact set such that $(x_n) \subset \Gamma$ and $x \in \Gamma$. Let $\varepsilon > 0$ and let $t_0 > 0$ be such that $s_n + t \leq t_0$, $n \in \mathbb{N}$. Using Assumption ($\mathcal{A}4$), we may find $r > 0$ such that

$$\sup_{z\in\Gamma} \mathbb{P}_x \left[\sup_{u\in[0,t_0]} \rho(X_s, x) \geq r \right] \leq \varepsilon.$$

Thus, setting $A := \{\rho(X_t, X_0) < r\}$, we get

$$\begin{aligned}
\sup_{z\in\Gamma} |\widetilde{\mathcal{P}}_t h(s_n, z) - \widetilde{\mathcal{P}}_t h(s,z)| &\leq \sup_{z\in\Gamma} \mathbb{E}_z \, |h(t+s_n, X_t) - h(t+s, X_t)| \\
&\leq \sup_{z\in\Gamma} \mathbb{E}_z \left[1_A \, |h(t+s_n, X_t) - h(t+s, X_t)| \right] \\
&\quad + \sup_{z\in\Gamma} \mathbb{E}_z \left[1_{A^c} \, |h(t+s_n, X_t) - h(t+s, X_t)| \right] \\
&\leq \sup_{z\in B(\Gamma,r)} |h(t+s_n, z) - h(t+s,z)| + 2\varepsilon \|h\|,
\end{aligned}$$

where $B(\Gamma, r) := \{z \in E \colon \rho(\Gamma, z) \leq r\}$. Noting that, from the continuity of h, we get

$$\sup_{z\in B(\Gamma,r)} |h(t+s_n, z) - h(t+s,z)| \to 0, \quad n \to \infty,$$

and recalling (3.3.33), we get $\widetilde{\mathcal{P}}_t \mathcal{C}_b(\widetilde{E}) \subset \mathcal{C}_b(\widetilde{E})$.

To show the properties of $\widetilde{M}_{\widetilde{\Gamma}}(t,r)$, it is enough to consider $\widetilde{\Gamma} := [0, T_0] \times \Gamma$ for some $T_0 > 0$ and a compact set $\Gamma \subset E$. Then, for any $t, r > 0$, we get

$$\begin{aligned}
\widetilde{M}_{\widetilde{\Gamma}}(t,r) &= \sup_{\substack{t'\in[0,T_0] \\ x\in\Gamma}} \mathbb{P}_x \left[\sup_{s\in[0,t]} \widetilde{\rho}((t'+s, X_s), (t', x)) \geq r \right] \\
&= \sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t]} (|s| + \rho(X_s, x)) \geq r \right] \\
&\leq \sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t]} \rho(X_s, x) \geq r - t \right].
\end{aligned}$$

In particular, for $r_0 > 0$ and any $t < \frac{1}{2}r_0$, we get

$$\widetilde{M}_{\widetilde{\Gamma}}(t, r_0) \leq \sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t]} \rho(X_s, x) \geq \frac{1}{2} r_0 \right].$$

Thus, using Assumption ($\mathcal{A}4$), we get $\lim_{t\to 0} \widetilde{M}_{\widetilde{\Gamma}}(t, r_0) = 0$. Similarly, for $t_0 > 0$ and any $r = t_0 + r' > t_0$, we get

$$\widetilde{M}_{\widetilde{\Gamma}}(t_0, r) \leq \sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t_0]} \rho(X_s, x) \geq r' \right].$$

Thus, letting $r' \to \infty$, from Assumption ($\mathcal{A}4$), we get $\lim_{r\to\infty} \widetilde{M}_{\widetilde{\Gamma}}(t_0, r) = 0$. Hence, the version of Assumption ($\mathcal{A}4$) is satisfied for the space-time process $\widetilde{X}$. Also, Assumptions ($\mathcal{A}1$)–($\mathcal{A}3$) are satisfied due to the boundedness of $\widetilde{G}$; see also the discussion at the beginning of this section. Consequently, Theorem 3.3.2 can be applied to (3.3.30) and (3.3.31).

In the setting of the space-time process, for any $T, t \geq 0$ and $x \in E$, directly from the definition of $\widetilde{X}$, we get

$$\underline{w}_T(t, x) = \inf_{\tau\leq T} \ln \widetilde{\mathbb{E}}_{(t,x)} \left[\exp\left(\int_0^\tau \widetilde{g}(\widetilde{X}_s) ds + 1_{\{\tau<T\}} \widetilde{G}(\widetilde{X}_\tau) \right) \right],$$
$$\overline{w}_T(t, x) = \inf_{\tau\leq T} \ln \widetilde{\mathbb{E}}_{(t,x)} \left[\exp\left(\int_0^\tau \widetilde{g}(\widetilde{X}_s) ds + \widetilde{G}(\widetilde{X}_\tau) \right) \right].$$

Thus, using Theorem 3.3.2, we get that the maps $(T, t, x) \mapsto \underline{w}_T(t, x)$ and $(T, t, x) \mapsto \overline{w}_T(t, x)$ are jointly continuous. Also, for any $t \geq 0$ and $x \in E$, the stopping times

$$\widetilde{\underline{\tau}}_T := \inf\left\{ s \geq 0 : \underline{w}_{T-s}(\widetilde{X}_s) = \widetilde{G}(\widetilde{X}_s) \right\} \wedge T,$$
$$\widetilde{\overline{\tau}}_T := \inf\left\{ s \geq 0 : \overline{w}_{T-s}(\widetilde{X}_s) = \widetilde{G}(\widetilde{X}_s) \right\}$$

are optimal for $\underline{w}_T(t, x)$ and $\overline{w}_T(t, x)$, respectively. Noting that under $\widetilde{\mathbb{P}}_{(t,x)}$, $t \geq 0$, $x \in E$, we get $\widetilde{\underline{\tau}}_T = \underline{\tau}_T(t)$ and $\widetilde{\overline{\tau}}_T = \overline{\tau}_T(t)$, we conclude the proof of the optimality of $\underline{\tau}_T(t)$ and $\overline{\tau}_T(t)$. Using a similar argument and the martingale characterisation from Theorem 3.3.2, we also get the martingale properties of $\underline{z}_{T,t}$ and $\overline{z}_{T,t}$, which concludes the proof. $\square$

3.4 Infinite time horizon continuous time stopping

In this section, we consider the value functions of the infinite time horizon continuous time optimal stopping problems given by

$$\underline{w}(x) := \inf_{\tau\in\mathcal{T}_x} \ln \mathbb{E}_x\left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau)\right)\right], \quad x\in E, \tag{3.4.1}$$

$$\overline{w}(x) := \inf_{\tau\in\mathcal{T}_{x,b}} \ln \mathbb{E}_x\left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau)\right)\right], \quad x\in E. \tag{3.4.2}$$

Assuming ($\mathcal{A}1$)–($\mathcal{A}4$), we show certain regularity properties of the maps $\underline{w}$ and $\overline{w}$. This includes semi-continuity and finite time horizon approximation schemes. Also, we provide formulae for optimal stopping times and link the value functions to a suitable form of the Bellman equation.

3.4.1 Continuous time Bellman equation

The analysis in this section is based on properties of maps $w\in\mathcal{C}^+(E)$ which are solutions to the continuous time Bellman equation given by

$$e^{w(x)} = \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{\int_0^{\tau\wedge t} g(X_s)ds + 1_{\{\tau<t\}}G(X_\tau)+1_{\{\tau\geq t\}}w(X_t)}\right], \quad x\in E,\, t\geq 0. \tag{3.4.3}$$

We show that the maps given by (3.4.1) and (3.4.2) are minimal and maximal solutions to this equation, respectively. Also, we show that, in the bounded case, Equation (3.4.3) admits a unique solution which facilitates the proof of the continuity properties of the maps $\underline{w}$ and $\overline{w}$.

Before we proceed, let us clarify a terminological issue. Following the discussion in Section 3.2.1, Equation (3.4.3) should be called the continuous time dynamic programming principle. In fact, in the continuous time setting, the Bellman equation is typically associated with a specific functional equation related to the infinitesimal generator of the process. Nevertheless, to simplify the narrative, we refer to (3.4.3) as the continuous time Bellman equation.

In Proposition 3.4.1 we study the properties of continuous solutions to the Bellman equation (3.4.3). The first part of the proposition may be seen as a continuous time version of Lemma 3.2.1. Note that here, in contrast to the discrete time case, we additionally require the continuity of w. The second part of the proposition gives a characterisation of the situation when a continuous solution to the Bellman equation could be expressed as an expectation of the stopped value process.

Proposition 3.4.1. *Let $w \in \mathcal{C}^+(E)$ be a solution to (3.4.3). Also, let us define*

$$\tau_w := \inf\{t \geq 0 : w(X_t) = G(X_t)\}. \tag{3.4.4}$$

Then:

(1) *The infimum in (3.4.3) is attained for the stopping time τ_w, i.e. for any $x \in E$ and $T \geq 0$, we get*

$$e^{w(x)} = \mathbb{E}_x \left[e^{\int_0^{\tau_w \wedge T} g(X_s)ds + 1_{\{\tau_w < T\}} G(X_{\tau_w}) + 1_{\{\tau_w \geq T\}} w(X_T)} \right]. \tag{3.4.5}$$

Moreover, for any $x \in E$, the process

$$z_w(t) := \exp\left(\int_0^t g(X_s)ds + w(X_t) \right), \quad t \geq 0, \tag{3.4.6}$$

is a $\mathbb{P}_x$-submartingale and $(z_w(\tau_w \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale.

(2) *We have*

$$e^{w(x)} = \mathbb{E}_x \left[e^{\int_0^{\tau_w} g(X_s)ds + G(X_{\tau_w})} \right], \quad x \in E$$

if and only if

$$\lim_{T\to\infty} \mathbb{E}_x \left[1_{\{\tau_w \geq T\}} e^{\int_0^T g(X_s)ds + w(X_T)} \right] = 0, \quad x \in E.$$

Proof. For transparency, we prove the claims point by point.

Proof of (1). For any $T \geq 0$ and $x \in E$, let us define

$$e^{w_T(x)} := \inf_{\tau \leq T} \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + 1_{\{\tau < T\}} G(X_\tau) + 1_{\{\tau = T\}} w(X_T)} \right]$$

and note that, by (3.4.3), in fact we have $w_T \equiv w$ for any $T \geq 0$. In particular, using the continuity of $x \mapsto w(x)$, we get that the map $(T, x) \mapsto w_T(x)$ is jointly continuous. Hence, using Proposition A.2.8, we get that the stopping time

$$\tau_T := \inf\{t \geq 0 : w_{T-t}(X_t) = G(X_t)\} \wedge T = \tau_w \wedge T \tag{3.4.7}$$

is optimal for w_T. Thus, for any $x \in E$ and $T \geq 0$, we get

$$\begin{aligned} e^{w(x)} = e^{w_T(x)} &= \mathbb{E}_x \left[e^{\int_0^{\tau_T} g(X_s)ds + 1_{\{\tau_T < T\}} G(X_{\tau_T}) + 1_{\{\tau_T = T\}} w(X_T)} \right] \\ &= \mathbb{E}_x \left[e^{\int_0^{\tau_w \wedge T} g(X_s)ds + 1_{\{\tau_w < T\}} G(X_{\tau_w}) + 1_{\{\tau_w \geq T\}} w(X_T)} \right], \end{aligned}$$

and (3.4.5) holds. Also, using again Proposition A.2.8, we get that $(z_w(t))$, $t \geq 0$, is a submartingale, and $(z_w(\tau_T \wedge t))$, $t \geq 0$, is a martingale.

Proof of (2). Using the first part, for any $x \in E$ and $T \geq 0$, we get

$$\begin{aligned} e^{w(x)} &= \mathbb{E}_x \left[e^{\int_0^{\tau_w \wedge T} g(X_s)ds + 1_{\{\tau_w < T\}} G(X_{\tau_w}) + 1_{\{\tau_w \geq T\}} w(X_T)} \right] \\ &= \mathbb{E}_x \left[1_{\{\tau_w < T\}} e^{\int_0^{\tau_w \wedge T} g(X_s)ds + G(X_{\tau_w})} + 1_{\{\tau_w \geq T\}} e^{\int_0^{\tau_w \wedge T} g(X_s)ds + w(X_T)} \right]. \end{aligned} \tag{3.4.8}$$

Thus, recalling that $g(\cdot) \geq c > 0$ and using Fatou's lemma, for any $x \in E$, we get

$$\mathbb{E}_x \left[e^{\tau_w c} \right] \leq \mathbb{E}_x \left[\liminf_{T \to \infty} e^{(\tau_w \wedge T)c} \right] \leq \mathbb{E}_x \left[\liminf_{T \to \infty} e^{\int_0^{\tau_w \wedge T} g(X_s)ds} \right] \leq e^{w(x)} < \infty,$$

and, in particular, we get $\mathbb{P}_x[\tau_w < \infty] = 1$, $x \in E$. Thus, letting $T \to \infty$ in (3.4.8), we get

$$\begin{aligned} e^{w(x)} &= \lim_{T \to \infty} \mathbb{E}_x \left[1_{\{\tau_w < T\}} e^{\int_0^{\tau_w} g(X_s)ds + G(X_{\tau_w})} + 1_{\{\tau_w \geq T\}} e^{\int_0^{T} g(X_s)ds + w(X_T)} \right] \\ &= \mathbb{E}_x \left[e^{\int_0^{\tau_w} g(X_s)ds + G(X_{\tau_w})} \right] + \lim_{T \to \infty} \mathbb{E}_x \left[1_{\{\tau_w \geq T\}} e^{\int_0^{T} g(X_s)ds + w(X_T)} \right], \end{aligned}$$

where the second equality follows from the monotone convergence theorem. This concludes the proof. □

3.4.2 Solution to the problem

In this section, by analogy to Section 3.2, we show that the maps $\underline{w}$ and $\overline{w}$ are minimal and maximal solutions to (3.4.3), respectively. Also, we discuss the conditions that guarantee the uniqueness of a solution to the Bellman equation. This facilitates the analysis of the regularity properties of $\underline{w}$ and $\overline{w}$.

We start with noting that (3.4.3) may be expressed in the operator form as

$$w(x) = S_t w(x), \quad w \in \mathcal{M}^+(E),\ t \geq 0,\ x \in E, \tag{3.4.9}$$

where the operator $S_t \colon \mathcal{M}^+(E) \to \mathcal{M}^+(E)$, $t \geq 0$, is given by

$$S_t h(x) := \inf_{\tau \in \mathcal{T}} \ln \mathbb{E}_x \left[e^{\int_0^{\tau \wedge t} g(X_s)ds + 1_{\{\tau < t\}} G(X_\tau) + 1_{\{\tau \geq t\}} h(X_t)} \right] \tag{3.4.10}$$

with $h \in \mathcal{M}^+(E)$ and $x \in E$. By analogy to (3.2.10) and (3.2.11), in order to approximate solutions to (3.4.3), we iterate S_t on the suitable lower and upper bounds. More specifically, for any $t \geq 0$, we recursively define

$$\underline{v}_0^t(x) := 0, \qquad \underline{v}_{n+1}^t(x) := S_t \underline{v}_n^t(x), \qquad n \in \mathbb{N},\ x \in E, \tag{3.4.11}$$
$$\overline{v}_0^t(x) := G(x), \qquad \overline{v}_{n+1}^t(x) := S_t \overline{v}_n^t(x), \qquad n \in \mathbb{N},\ x \in E. \tag{3.4.12}$$

We start with linking the iterates of the operator S_t to the finite time horizon optimal stopping value functions.

Proposition 3.4.2. *For any $t \geq 0$ and $n \in \mathbb{N}$, let the maps $\underline{v}_n^t$ and $\overline{v}_n^t$ be given by (3.4.11) and (3.4.12), respectively. Also, for any $T \geq 0$, let the maps $\underline{w}_T$ and $\overline{w}_T$ be given by (3.3.1) and (3.3.2), respectively. Then, we get*

$$\underline{v}_n^t(x) \equiv \underline{w}_{nt}(x) \quad \text{and} \quad \overline{v}_n^t(x) \equiv \overline{w}_{nt}(x), \quad n \in \mathbb{N},\ t \geq 0,\ x \in E.$$

Proof. We present the proof only for $\underline{v}_n^t$; the argument for $\overline{v}_n^t$ is similar and is omitted for brevity. Also, for notational convenience, we set $t = 1$; the general case follows the same logic.

We proceed by induction. The claim for $n = 0$ follows directly from the definition. Let us assume that, for some $n \in \mathbb{N}$, we get $\underline{v}_n^1 \equiv \underline{w}_n$. Define the process

$$\underline{z}_{n+1}(t) := e^{\int_0^{t\wedge(n+1)} g(X_s)ds + \underline{w}_{n+1-t\wedge(n+1)}(X_{t\wedge(n+1)})}, \quad t \geq 0.$$

Using Theorem 3.3.2 and Doob's optional stopping theorem, for any stopping time $\tau \in \mathcal{T}$, we get that the process $(\underline{z}_{n+1}(\tau \wedge t))$, $t \geq 0$, is a submartingale. In particular, for any $x \in E$, we get $\mathbb{E}_x[\underline{z}_{n+1}(0)] \leq \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left[\underline{z}_{n+1}(\tau \wedge 1)\right]$. Then, recalling that $\underline{w}_T(x) \leq G(x)$, $T \geq 0$, $x \in E$, we get

$$\begin{aligned} e^{\underline{w}_{n+1}(x)} &= \mathbb{E}_x[\underline{z}_{n+1}(0)] \\ &\leq \inf_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{\int_0^{\tau\wedge 1} g(X_s)ds + \underline{w}_{n+1-\tau\wedge 1}(X_{\tau\wedge 1})}\right] \\ &\leq \inf_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{\int_0^{\tau\wedge 1} g(X_s)ds + 1_{\{\tau<1\}}G(X_\tau) + 1_{\{\tau\geq 1\}}\underline{w}_n(X_1)}\right]. \end{aligned} \tag{3.4.13}$$

Recall that, by Theorem 3.3.2, the process $(\underline{z}_{n+1}(\underline{\tau}_{n+1} \wedge t))$, $t \geq 0$, is a martingale, where $\underline{\tau}_{n+1} := \inf\{t \geq 0 : \underline{w}_{n+1-t}(X_t) = G(X_t)\} \wedge (n+1)$. Also, using the continuity of $(T, x) \mapsto \underline{w}_T(x)$ and the right-continuity of X, on the event $\{\underline{\tau}_{n+1} < n+1\}$, we get $\underline{w}_{n+1-\underline{\tau}_{n+1}}(X_{\underline{\tau}_{n+1}}) = G(X_{\underline{\tau}_{n+1}})$. Thus, for any $x \in E$, we get

$$\begin{aligned} e^{\underline{w}_{n+1}(x)} &= \mathbb{E}\left[e^{\int_0^{\underline{\tau}_{n+1}\wedge 1} g(X_s)ds + \underline{w}_{n+1-\underline{\tau}_{n+1}\wedge 1}(X_{\underline{\tau}_{n+1}\wedge 1})}\right]. \\ &= \mathbb{E}\left[e^{\int_0^{\underline{\tau}_{n+1}\wedge 1} g(X_s)ds + 1_{\{\underline{\tau}_{n+1}<1\}}G(X_{\underline{\tau}_{n+1}}) + 1_{\{\underline{\tau}_{n+1}\geq 1\}}\underline{w}_n(X_1)}\right]. \end{aligned}$$

Combining this with (3.4.13) and using the induction assumption, for any $x \in E$, we get

$$\begin{aligned} e^{\underline{w}_{n+1}(x)} &= \inf_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{\int_0^{\tau\wedge 1} g(X_s)ds + 1_{\{\tau<1\}}G(X_\tau)+1_{\{\tau\geq 1\}}\underline{w}_n(X_1)}\right] \\ &= e^{S_1\underline{w}_n(x)} = e^{S_1\underline{v}_n^1(x)} = e^{\underline{v}_{n+1}^1(x)}, \end{aligned}$$

which concludes the proof. □

In the following theorem, we characterise the solutions to the Bellman equation. In particular, we get that $\underline{w}$ and $\overline{w}$ are minimal and maximal solutions to (3.4.3), respectively. This may be seen as a continuous time version of Theorem 3.2.6.

Theorem 3.4.3. *Let the maps $\underline{w}$ and $\overline{w}$ be given by (3.4.1) and (3.4.2), respectively. Also, for any $T \geq 0$, let the maps $\underline{w}_T$ and $\overline{w}_T$ are given by (3.3.1) and (3.3.2), respectively. Then:*

(1) *For any $x \in E$, we get*

$$\underline{w}(x) = \lim_{T\to\infty} \underline{w}_T(x) \quad \text{and} \quad \overline{w}(x) = \lim_{T\to\infty} \overline{w}_T(x).$$

Also, $\underline{w}$ is lower semi-continuous and $\overline{w}$ is upper semi-continuous.

(2) *The maps $\underline{w}$ and $\overline{w}$ are solutions to (3.4.3).*

(3) *For any solution $w \in \mathcal{M}^+(E)$ to the Bellman equation (3.4.3), we get $\underline{w}(\cdot) \leq w(\cdot) \leq \overline{w}(\cdot)$.*

Proof. For transparency, we prove the claim point by point.

Proof of (1). The proof that $\overline{w}(x) = \lim_{T\to\infty} \overline{w}_T(x)$, $x \in E$, follows the lines of the argument in Step 2 in the proof of Theorem 3.2.6 and is omitted for brevity. Now we show that $\underline{w}(x) = \lim_{T\to\infty} \underline{w}_T(x)$, $x \in E$.

Recalling the non-negativity of g and G, for any $T \geq$ and $x \in E$, we get

$$\begin{aligned} e^{\underline{w}_T(x)} &= \inf_{\tau\in\mathcal{T}_x} \mathbb{E}_x\left[e^{\int_0^{\tau\wedge T} g(X_s)ds + 1_{\{\tau<T\}}G(X_\tau)}\right] \\ &\leq \inf_{\tau\in\mathcal{T}_x} \mathbb{E}_x\left[e^{\int_0^{\tau} g(X_s)ds + G(X_\tau)}\right] = e^{\underline{w}(x)}. \end{aligned}$$

Thus, we get $\lim_{T\to\infty} \underline{w}_T(x) \leq \underline{w}(x)$, $x \in E$. Let us now show the reverse inequality. For any $T \geq 0$, define

$$\hat{\underline{\tau}}_T := \inf\left\{t \geq 0 : \underline{w}_{T-t}(X_t) = G(X_t)\right\} \tag{3.4.14}$$

and note that $\underline{\tau}_T = \widehat{\underline{\tau}}_T \wedge T$, where $\underline{\tau}_T$ is an optimal stopping time for $\underline{w}_T$ given by (3.3.9). Recalling Theorem 3.3.2, for any $x \in E$, we get that the sequence $(\underline{w}_n(x))_{n\in\mathbb{N}}$ is increasing and bounded from above by $G(x)$. Thus, we get $\widehat{\underline{\tau}}_{n+1} \leq \widehat{\underline{\tau}}_n$, $n \in \mathbb{N}$. We show that, for any $x \in E$ and $\mathbb{P}_x$ almost all $\omega \in \Omega$, starting from some $n \in \mathbb{N}$ (depending on ω), the sequence $(\underline{\tau}_n(\omega))$, $n \in \mathbb{N}$, is decreasing. Note that, from the monotonicity of $(\widehat{\underline{\tau}}_n)$, for any $n \in \mathbb{N}$, on the event $\{\underline{\tau}_n < n\}$, we get

$$\widehat{\underline{\tau}}_{n+1} \leq \widehat{\underline{\tau}}_n = \underline{\tau}_n < n, \tag{3.4.15}$$

thus $\widehat{\underline{\tau}}_{n+1} = \underline{\tau}_{n+1}$, and consequently $\underline{\tau}_{n+1} \leq \underline{\tau}_n$ on $\{\underline{\tau}_n < n\}$. Moreover, recalling that $g(\cdot) \geq c > 0$ and $G(\cdot) \geq 0$, for any $n \in \mathbb{N}$ and $x \in E$, we get

$$e^{G(x)} \geq e^{\underline{w}_n(x)} = \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}_n} g(X_s)ds + 1_{\{\underline{\tau}_n<n\}}G(X_{\underline{\tau}_n})}\right] \geq \mathbb{E}_x\left[1_{\{\underline{\tau}_n=n\}}\right]e^{cn}. \tag{3.4.16}$$

Consequently, for any $x \in E$, we get $\sum_{n=1}^{\infty}\mathbb{P}_x\left[\underline{\tau}_n = n\right] \leq \sum_{n=0}^{\infty}\frac{e^{G(x)}}{e^{cn}} < \infty$. Thus, using the Borel-Cantelli lemma, we get $\mathbb{P}_x\left[\bigcup_{n=0}^{\infty}\{\underline{\tau}_n < n\}\right] = 1$, $x \in E$. Hence, recalling (3.4.15) and the following discussion, we get that, for any $\omega \in \bigcup_{n=0}^{\infty}\{\underline{\tau}_n < n\}$, starting from some $n \in \mathbb{N}$ (depending on ω), we get $\underline{\tau}_{n+k+1}(\omega) \leq \underline{\tau}_{n+k}(\omega)$, $k \in \mathbb{N}$. Consequently, the stopping time

$$\widehat{\tau} := \lim_{n\to\infty}\underline{\tau}_n \tag{3.4.17}$$

is well defined and $\widehat{\tau} \in \mathcal{T}_x$ for any $x \in E$. Also, using the right-continuity of X, we get $\lim_{n\to\infty} 1_{\{\underline{\tau}_n<n\}}G(X_{\underline{\tau}_n}) = G(X_{\widehat{\tau}})$ $\mathbb{P}_x$ a.s. Thus, using Fatou's lemma, for any $x \in E$, we get

$$\begin{aligned} e^{\underline{w}(x)} \leq \mathbb{E}_x\left[e^{\int_0^{\widehat{\tau}} g(X_s)ds + G(X_{\widehat{\tau}})}\right] &= \mathbb{E}_x\left[\lim_{n\to\infty}\left(e^{\int_0^{\underline{\tau}_n} g(X_s)ds + 1_{\{\underline{\tau}_n<n\}}G(X_{\underline{\tau}_n})}\right)\right] \\ &\leq \lim_{n\to\infty} e^{\underline{w}_n(x)} \end{aligned} \tag{3.4.18}$$

and, consequently, we get $\lim_{n\to\infty}\underline{w}_n(x) = \underline{w}(x)$, $x \in E$. In fact, using the monotonicity of $T \mapsto \underline{w}_T(x)$, $x \in E$, we also get $\lim_{T\to\infty}\underline{w}_T(x) = \underline{w}(x)$, $x \in E$.

Finally, using Theorem 3.3.2, we get that the map $x \mapsto \underline{w}(x)$ is the increasing limit of the continuous functions $x \mapsto \underline{w}_n(x)$. Thus, we get that $\underline{w}$ is lower semi-continuous. Similarly, noting that the map $x \mapsto \overline{w}(x)$ is the decreasing limit of the continuous functions $x \mapsto \overline{w}_n(x)$, we get that $\overline{w}$ is upper semi-continuous.

Proof of (2). First, we prove that $\underline{w}$ satisfies (3.4.3). Let us define the process

$$\underline{z}(t) := e^{\int_0^t g(X_s)ds + \underline{w}(X_t)}, \quad t \geq 0. \tag{3.4.19}$$

We show that $(\underline{z}(t))$, $t \geq 0$, is a submartingale. From Theorem 3.3.2, using the submartingale property of the process $(\underline{z}_T(t))$ from (3.3.10), for any $T, t, h \geq 0$ and $x \in E$, we get

$$e^{\int_0^{t\wedge T} g(X_s)ds + \underline{w}_{T-t\wedge T}(X_{t\wedge T})} \leq \mathbb{E}_x\left[e^{\int_0^{(t+h)\wedge T} g(X_s)ds + \underline{w}_{T-(t+h)\wedge T}(X_{(t+h)\wedge T})} \middle| \mathcal{F}_t \right].$$

Thus, recalling the monotonicity of $T \mapsto \underline{w}_T(x)$, $x \in E$, and letting $T \to \infty$, for any $t, h \geq 0$ and $x \in E$, we get

$$\underline{z}(t) = e^{\int_0^{t} g(X_s)ds + \underline{w}(X_t)} \leq \mathbb{E}_x\left[e^{\int_0^{t+h} g(X_s)ds + \underline{w}(X_{t+h})} \middle| \mathcal{F}_t \right] = \mathbb{E}_x\left[\underline{z}(t+h) | \mathcal{F}_t \right], \tag{3.4.20}$$

which shows the submartingale property of $(\underline{z}(t))$, $t \geq 0$.

Next, using Doob's optional stopping theorem and the fact that $\underline{w} \leq G$, for any $t \geq 0$ and $x \in E$, we get

$$\begin{aligned} e^{\underline{w}(x)} = \mathbb{E}_x\left[\underline{z}(0)\right] &\leq \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left[\underline{z}(\tau\wedge t)\right] \\ &\leq \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{\int_0^{\tau\wedge t} g(X_s)ds + 1_{\{\tau<t\}}G(X_\tau) + 1_{\{\tau\geq t\}}\underline{w}(X_t)} \right]. \end{aligned} \tag{3.4.21}$$

To conclude the proof that $\underline{w}$ satisfies (3.4.3), we show that, for any $t \geq 0$ and $x \in E$, we get

$$e^{\underline{w}(x)} = \mathbb{E}_x\left[e^{\int_0^{\hat{\tau}\wedge t} g(X_s)ds + 1_{\{\hat{\tau}<t\}}G(X_{\hat{\tau}}) + 1_{\{\hat{\tau}\geq t\}}\underline{w}(X_t)} \right], \tag{3.4.22}$$

where the stopping time $\hat{\tau}$ is given by (3.4.17). From Theorem 3.3.2, using the martingale property of $(\underline{z}_T(t \wedge \underline{\tau}_T))$, for any $t \geq 0$, $T \geq t$, and $x \in E$, we get

$$\begin{aligned} e^{\underline{w}_T(x)} = \mathbb{E}_x[\underline{z}_T(0)] &= \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}_T\wedge t} g(X_s)ds + \underline{w}_{T-\underline{\tau}_T\wedge t}(X_{\underline{\tau}_T\wedge t})} \right] \\ &= \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}_T\wedge t} g(X_s)ds + 1_{\{\underline{\tau}_T<t\}}G(X_{\underline{\tau}_T}) + 1_{\{\underline{\tau}_T\geq t\}}\underline{w}_{T-t}(X_t)} \right]. \end{aligned}$$

Thus, using the right-continuity of X, recalling Assumption ($\mathcal{A}2$), and letting $T \to \infty$, we get (3.4.22). Combining this with (3.4.21), for any $t \geq 0$ and $x \in E$, we get

$$e^{\underline{w}(x)} = \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{\int_0^{\tau\wedge t} g(X_s)ds + 1_{\{\tau<t\}}G(X_\tau) + 1_{\{\tau\geq t\}}\underline{w}(X_t)} \right].$$

Thus, we get that $\underline{w}$ satisfies (3.4.3).

Second, we prove that $\overline{w}$ is also a solution to (3.4.3). Combining Theorem 3.3.2, the first part of this theorem, and Proposition 3.4.2, for any $t \geq 0$, we get $\overline{w}(x) = \lim_{T\to\infty} \overline{w}_T(x) = \inf_{n\in\mathbb{N}} \overline{v}^t_{n+1}(x)$, $x \in E$, where $\overline{w}_T$ and $\overline{v}^t_{n+1}$ are given (3.3.2) and (3.4.12), respectively. Thus, using the monotone convergence theorem, for any $t \geq 0$ and $x \in E$, we get

$$\begin{aligned}
e^{\overline{w}(x)} &= \inf_{n\in\mathbb{N}} e^{\overline{v}^t_{n+1}(x)} \\
&= \inf_{\tau\in\mathcal{T}} \inf_{n\in\mathbb{N}} \mathbb{E}_x \left[e^{\int_0^{\tau\wedge t} g(X_s)ds + 1_{\{\tau<t\}}G(X_\tau) + 1_{\{\tau\geq t\}}\overline{v}^t_n(X_1)} \right] \\
&= \inf_{\tau\in\mathcal{T}} \mathbb{E}_x \left[e^{\int_0^{\tau\wedge t} g(X_s)ds + 1_{\{\tau<t\}}G(X_\tau) + 1_{\{\tau\geq t\}}\overline{w}(X_t)} \right] = e^{S_t\overline{w}(x)},
\end{aligned}$$

thus $\overline{w}$ is a solution to (3.4.3).

Proof of (3). Recall that, by (3.4.9), if w is a solution to (3.4.3), then $w(x) = S_t w(x)$, $x \in E$, $t \geq 0$. Thus, recalling (3.4.11) and (3.4.12), and using the fact that $0 \leq w(\cdot) \leq G(\cdot)$, inductively we get $\underline{v}^t_n(x) \leq w(x) \leq \overline{v}^t_n(x)$ for any $t \geq 0$, $n \in \mathbb{N}$, and $x \in E$. Thus, using Proposition 3.4.2, we get $\underline{w}_{nt}(x) \leq w(x) \leq \overline{w}_{nt}(x)$ for any $t \geq 0$, $n \in \mathbb{N}$, and $x \in E$. Hence, letting $n \to \infty$ and using the first part of this theorem, we conclude the proof. □

Remark 3.4.4. Note that, from the proof of Theorem 3.4.3, we get that, for any $x \in E$, the stopping time $\widehat{\tau} \in \mathcal{T}_x$, given by (3.4.17), is optimal for $\underline{w}(x)$; see (3.4.18) and the following discussion. However, it should be noted that $\widehat{\tau}$ is defined as the limit of $\underline{\tau}_n$, which makes it difficult to analyse. In Proposition 3.4.5 we find an explicit formula for another optimal stopping time for $\underline{w}$, provided that the map $x \mapsto \underline{w}(x)$ is continuous. ◊

Now, we show the closed-form formula for an optimal stopping time for $\underline{w}$ under the continuity assumption.

Proposition 3.4.5. *Let the map* $\underline{w}$ *be given by* (3.4.1) *and assume that* $x \mapsto \underline{w}(x)$ *is continuous. Then, for any* $x \in E$*, the stopping time*

$$\underline{\tau} := \inf\{t \geq 0 : \underline{w}(X_t) = G(X_t)\} \in \mathcal{T}_x \tag{3.4.23}$$

is optimal for $\underline{w}(x)$*.*

Proof. Using Theorem 3.4.3, we get that $\underline{w}$ satisfies (3.4.3). Thus, from Proposition 3.4.1, we get that the process $(z_{\underline{w}}(t\wedge\underline{\tau}))$, $t \geq 0$, given by (3.4.6), is a martingale. Hence, using the fact that $g(\cdot) \geq c > 0$, the non-negativity of $\underline{w}$, the quasi left-continuity of X, and Fatou's lemma, for any $x \in E$, we get

$$\begin{aligned}
\mathbb{E}_x\left[e^{c\underline{\tau}}\right] &\leq \mathbb{E}_x\left[\liminf_{t\to\infty} z_{\underline{w}}(t\wedge\underline{\tau})\right] \\
&\leq \liminf_{t\to\infty} \mathbb{E}_x\left[z_{\underline{w}}(t\wedge\underline{\tau})\right] \\
&= \mathbb{E}_x\left[z_{\underline{w}}(0)\right] = e^{\underline{w}(x)} \leq e^{G(x)} < \infty.
\end{aligned} \tag{3.4.24}$$

In particular, we get $\underline{\tau} \in \mathcal{T}_x$ and, from the continuity of $\underline{w}$ and the right-continuity of X, we get $\underline{w}(X_{\underline{\tau}}) = G(X_{\underline{\tau}})$. Thus, using the quasi left-continuity of X, Fatou's lemma, and the martingale property of $(z_{\underline{w}}(t \wedge \underline{\tau}))$, $t \geq 0$, we get

$$\begin{aligned}
e^{\underline{w}(x)} &\leq \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}} g(X_s)ds + G(X_{\underline{\tau}})}\right] \\
&= \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}} g(X_s)ds + \underline{w}(X_{\underline{\tau}})}\right] \\
&= \mathbb{E}_x\left[\lim_{t\to\infty} e^{\int_0^{\underline{\tau}\wedge t} g(X_s)ds + \underline{w}(X_{\underline{\tau}\wedge t})}\right] \\
&\leq \liminf_{t\to\infty} \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}\wedge t} g(X_s)ds + \underline{w}(X_{\underline{\tau}\wedge t})}\right] = e^{\underline{w}(x)}, \quad x \in E,
\end{aligned}$$

which concludes the proof. $\square$

Remark 3.4.6. Recall that, by Remark 3.4.4, the stopping time $\hat{\tau}$ from (3.4.17) is also optimal for $\underline{w}$. It should be noted that the optimal stopping time $\underline{\tau}$ from (3.4.23) leads to earlier stopping compared to $\hat{\tau}$. Indeed, noting that $\underline{w}(\cdot) \geq \underline{w}_n(\cdot)$ for any $n \in \mathbb{N}$, on the event $\{\underline{\tau}_n < n\}$, we get

$$\underline{w}(X_{\underline{\tau}_n}) \geq \underline{w}_{n-\underline{\tau}_n}(X_{\underline{\tau}_n}) \geq G(X_{\underline{\tau}_n}),$$

hence we get $\underline{\tau} \leq \underline{\tau}_n$ on $\{\underline{\tau}_n < n\}$. Recalling that from (3.4.16) we deduced $\mathbb{P}_x[\bigcup_{n=0}^{\infty}\{\underline{\tau}_n < n\}] = 1$, $x \in E$, and using the fact that $\hat{\tau} = \lim_{n\to\infty} \underline{\tau}_n$, we get $\underline{\tau} \leq \hat{\tau}$. $\Diamond$

Now, by analogy to Theorem 3.2.11, we formulate a sufficient condition for the equality $\underline{w} \equiv \overline{w}$. In particular, this gives uniqueness of a solution to (3.4.3). To simplify the notation, let us define the process

$$Z(t) := \exp\left(\int_0^t g(X_s)ds + G(X_t)\right), \quad t \geq 0. \tag{3.4.25}$$

Theorem 3.4.7. *Let the maps $\underline{w}$ and $\overline{w}$ be given by (3.4.1) and (3.4.2), respectively. Also, let $\hat{\tau}$ and $(Z(t))$, $t \geq 0$, be given by (3.4.17) and (3.4.25), respectively. Assume that, for any $x \in E$, the process $(Z(\hat{\tau} \wedge t))$, $t \geq 0$, is $\mathbb{P}_x$-uniformly integrable. Then:*

(1) *We get $\underline{w} \equiv \overline{w} \in \mathcal{C}^+(E)$.*

(2) *For any $x \in E$, the stopping time $\underline{\tau} \in \mathcal{T}_x$, given by (3.4.23), is optimal for $\underline{w}(x)$. Also, we get $\underline{\tau} = \lim_{T\to\infty} \overline{\tau}_T$, where $\overline{\tau}_T$ is given by (3.3.11).*

(3) *For any $x \in E$, the stopping time $\underline{\tau} \in \mathcal{T}_x$ given by (3.4.23) is optimal for $\overline{w}(x)$ in the sense of (3.1.11), i.e. we get*

$$\overline{w}(x) = \liminf_{T\to\infty} \ln \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}\wedge T} g(X_s)ds + G(X_{\underline{\tau}\wedge T})}\right], \quad x \in E. \tag{3.4.26}$$

Proof. For transparency, we prove the claims point by point.

Proof of (1). Recall that, by Remark 3.4.4, for any $x \in E$, the stopping time $\widehat{\tau}$, given by (3.4.17), is optimal for $\underline{w}(x)$, and we get $\widehat{\tau} \in \mathcal{T}_x$. Thus, using the quasi left-continuity of X and the uniform integrability assumption, for any $x \in E$, we get

$$e^{\overline{w}(x)} \leq \lim_{T\to\infty} \mathbb{E}_x \left[e^{\int_0^{\widehat{\tau}\wedge T} g(X_s)ds + G(X_{\widehat{\tau}\wedge T})} \right] = \mathbb{E}_x \left[e^{\int_0^{\widehat{\tau}} g(X_s)ds + G(X_{\widehat{\tau}})} \right] = e^{\underline{w}(x)}.$$

Recalling that we always have $\underline{w} \leq \overline{w}$, we conclude the proof of $\underline{w} \equiv \overline{w}$. Note that the continuity property follows from the lower semi-continuity of $\underline{w}$ and the upper semi-continuity of $\overline{w}$; see Theorem 3.4.3 for details.

Proof of (2). Note that the optimality of $\underline{\tau}$ follows from Proposition 3.4.5 and the fact that $\underline{w} \equiv \overline{w} \in \mathcal{C}^+(E)$. Let us now show that

$$\underline{\tau} = \lim_{T\to\infty} \overline{\tau}_T, \tag{3.4.27}$$

where $\overline{\tau}_T$ is given by (3.3.11). Recalling Theorem 3.3.2, we get that the map $T \mapsto \overline{\tau}_T$ is increasing, hence the limit $\overline{\tau} := \lim_{T\to\infty} \overline{\tau}_T$ is well defined. Also, recalling that, from (3.4.24), we get $\mathbb{P}_x [\underline{\tau} < \infty] = 1$, $x \in E$, and using the fact that $\underline{w} \equiv \overline{w} \leq \overline{w}_T$, $T \geq 0$, on the event $\{\underline{\tau} \leq T\}$, we get

$$\overline{w}_{T-\underline{\tau}}(X_{\underline{\tau}}) \geq \underline{w}(X_{\underline{\tau}}) \geq G(X_{\underline{\tau}}).$$

Thus, noting that $\underline{w}_S(\cdot) \leq G(\cdot)$, $S \geq 0$, we get $\overline{\tau}_T \leq \underline{\tau} \wedge T$. Hence, letting $T \to \infty$ in, we get $\overline{\tau} \leq \underline{\tau}$. In particular, we get $\mathbb{P}_x [\overline{\tau} < \infty] = 1$, $x \in E$. Also, recalling the joint continuity of $(T, x) \mapsto \overline{w}_T(x)$, we get

$$\overline{w}_{T-\overline{\tau}_T}(X_{\overline{\tau}_T}) = G(X_{\overline{\tau}_T}). \tag{3.4.28}$$

We show that this implies $\underline{w}(X_{\overline{\tau}}) = G(X_{\overline{\tau}})$ and consequently $\underline{\tau} \leq \overline{\tau}$. First, note that, from a.s. finiteness of $\overline{\tau}$, we get $(T - \overline{\tau}_T) \to \infty$ as $T \to \infty$. Second, note that, for any $T_n \to \infty$ and $x_n \to x$, we get

$$|\overline{w}_{T_n}(x_n) - \overline{w}(x)| \leq |\overline{w}_{T_n}(x_n) - \overline{w}(x_n)| + |\overline{w}(x_n) - \overline{w}(x)| \to 0, \quad n \to \infty;$$

this follows from Dini's theorem combined with the fact that $(\overline{w}_{T_n})$ is a sequence of the continuous functions converging monotonically to the continuous function $\overline{w}$. Thus, letting $T \to \infty$ in (3.4.28), we get $\overline{w}(X_{\overline{\tau}}) = G(X_{\overline{\tau}})$, which, combined with the fact that $\underline{w} \equiv \overline{w}$, concludes the proof of this part.

Proof of (3). To show (3.4.26) it is enough to prove the uniform integrability of $(Z(\underline{\tau} \wedge t))$, $t \geq 0$, and use the second part of this theorem. Recalling that $\underline{w}(\cdot) \geq \underline{w}_T(\cdot)$, $T \geq 0$, on the set $\{\underline{\tau}_T < T\}$, we get

$$\underline{w}(X_{\underline{\tau}_T}) \geq \underline{w}_{T-\underline{\tau}_T}(X_{\underline{\tau}_T}) \geq G(X_{\underline{\tau}_T}),$$

where $\underline{\tau}_T$ is given by (3.3.9). Thus, letting $T \to \infty$, using the continuity of $\underline{w}$, and recalling (3.4.17), we get $\underline{w}(X_{\widehat{\tau}}) \geq G(X_{\widehat{\tau}})$. In fact, noting that $\underline{w}(\cdot) \leq G(\cdot)$, we get $\underline{w}(X_{\widehat{\tau}}) = G(X_{\widehat{\tau}})$. Thus, recalling (3.4.23), we get

$$\underline{\tau} \leq \widehat{\tau}. \tag{3.4.29}$$

From Lemma 3.1.2 and the uniform integrability of $(Z(\widehat{\tau} \wedge t))$, $t \geq 0$, for any $x \in E$, we get $\liminf_{T\to\infty} \mathbb{E}_x \left[1_{\{\widehat{\tau}>T\}} Z(T)\right] = 0$. Hence, using (3.4.29), for any $x \in E$, we also get $\liminf_{T\to\infty} \mathbb{E}_x \left[1_{\{\underline{\tau}>T\}} Z(T)\right] = 0$ and, again by Lemma 3.1.2, we conclude the proof of the uniform integrability of $(Z(\underline{\tau} \wedge t))$, $t \geq 0$. Thus, recalling parts (1) and (2), for any $x \in E$, we get

$$e^{\overline{w}(x)} = e^{\underline{w}(x)} = \mathbb{E}_x \left[e^{\int_0^{\underline{\tau}} g(X_s)ds + G(X_{\underline{\tau}})} \right] = \lim_{T\to\infty} \mathbb{E}_x \left[e^{\int_0^{\underline{\tau}\wedge T} g(X_s)ds + G(X_{\underline{\tau}\wedge T})} \right],$$

which concludes the proof. $\square$

Remark 3.4.8. Recall that in (3.4.2), to define $\overline{w}$, we used the family of bounded stopping times. However, it is not clear if the stopping time $\underline{\tau}$ from (3.4.23) belongs to $\mathcal{T}_{x,b}$. Thus, the optimality of $\underline{\tau}$ in (3.4.26) is understood in the context of Proposition 3.1.1, where we provide an alternative formulation for $\overline{w}$. $\lozenge$

Remark 3.4.9. In Theorem 3.4.7, the continuity of $\underline{w}$ was a consequence of the identity $\underline{w} \equiv \overline{w}$. However, if we know in advance that $\underline{w}$ is continuous, we may obtain the results of Theorem 3.4.7 under weaker conditions. Namely, following the proof of Theorem 3.4.7, we can see that, assuming the continuity of $\underline{w}$, one may replace the uniform integrability of $(Z(\widehat{\tau} \wedge t))$, $t \geq 0$, by the uniform integrability of $(Z(\underline{\tau} \wedge t))$, $t \geq 0$, where $\underline{\tau}$ is given by (3.4.23). Note that, by Remark 3.4.6, the latter condition is less restrictive as $\underline{\tau} \leq \widehat{\tau}$. $\lozenge$

As in the discrete time case, if G is bounded, the uniform integrability condition is satisfied, cf. Proposition 3.2.13.

Proposition 3.4.10. *Let $G \in \mathcal{C}_b^+(E)$. Then, for any $x \in E$, the process $(Z(\widehat{\tau} \wedge t))$, $t \geq 0$, from Theorem 3.4.7, is $\mathbb{P}_x$-uniformly integrable.*

Proof. Recalling (3.4.15) and the following discussion, for any $x \in E$, we get that, for $\mathbb{P}_x$ almost all $\omega \in \Omega$, starting from some $n \in \mathbb{N}$ (depending on ω), the sequence $(\underline{\tau}_n(\omega))$ is decreasing. Thus, using the right-continuity of X, we get $G(X_{\widehat{\tau}}) = \lim_{n\to\infty} 1_{\{\underline{\tau}_n<n\}} G(X_{\underline{\tau}_n})$. Consequently, recalling the non-negativity of G, Theorem 3.3.2, and using Fatou's lemma, for any $x \in E$, we get

$$
\begin{aligned}
\mathbb{E}_x\left[e^{\int_0^{\widehat{\tau}} g(X_s)ds}\right] &\leq \mathbb{E}_x\left[e^{\int_0^{\widehat{\tau}} g(X_s)ds + G(X_{\widehat{\tau}})}\right]\\
&= \mathbb{E}_x\left[\lim_{n\to\infty} e^{\int_0^{\underline{\tau}_n} g(X_s)ds + 1_{\{\underline{\tau}_n<n\}} G(X_{\underline{\tau}_n})}\right]\\
&\leq \lim_{n\to\infty} \mathbb{E}_x\left[e^{\int_0^{\underline{\tau}_n} g(X_s)ds + 1_{\{\underline{\tau}_n<n\}} G(X_{\underline{\tau}_n})}\right]\\
&= \lim_{n\to\infty} e^{\underline{w}_n(x)} \leq e^{G(x)} < \infty.
\end{aligned}
$$

Combining this with the inequality $Z(\widehat{\tau} \wedge t) \leq e^{\int_0^{\widehat{\tau}} g(X_s)ds} e^{\|G\|}$, $t \geq 0$, we get that, by Lebesgue's dominated convergence theorem, the process $(Z(\widehat{\tau} \wedge t))$, $t \geq 0$, is uniformly integrable, which concludes the proof. ☐

For ease of reference, the properties of the optimal stopping problems with a bounded terminal cost function G are summarised in the following theorem.

Theorem 3.4.11. *Let $G \in \mathcal{C}_b^+(E)$ and let the maps $\underline{w}$ and $\overline{w}$ be given by (3.4.1) and (3.4.2), respectively. Also, let $w \in \mathcal{M}^+(E)$ be a solution to the Bellman equation (3.4.3). Then, we get*

$$\underline{w} \equiv w \equiv \overline{w} \in \mathcal{C}_b^+(E).$$

Also, for any $x \in E$, the stopping time

$$\underline{\tau} := \inf\{t \geq 0 : \underline{w}(X_t) = G(X_t)\} \in \mathcal{T}_x$$

is optimal for $\underline{w}(x)$. Moreover, for any $x \in E$, the process

$$z_{\underline{w}}(t) := \exp\left(\int_0^t g(X_s)ds + \underline{w}(X_t)\right), \quad t \geq 0,$$

is a $\mathbb{P}_x$-submartingale and $(z_{\underline{w}}(\underline{\tau} \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale.

Proof. Using Proposition 3.4.10 and Theorem 3.4.7, we get $\underline{w} \equiv \overline{w}$; note that this identity may also be deduced from Corollary 3.1.3. This, combined with Theorem 3.4.3, shows the uniqueness of a solution to the Bellman equation (3.4.3). Also, using the semi-continuity properties from Theorem 3.4.3, we get $\underline{w} \equiv \overline{w} \in \mathcal{C}_b^+(E)$. Finally, recalling Proposition 3.4.1 and Proposition 3.4.5, we conclude the proof. ☐

3.5 Approximation of optimal stopping problems

In this section, we consider two types of approximation schemes for the optimal stopping problems. Throughout this section, by $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ we denote a continuous time standard $\mathcal{C}_b$-Feller–Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Also, we assume $(\mathcal{A}1)$–$(\mathcal{A}4)$.

3.5.1 Approximation with a bounded terminal cost function

In this section, we show that the map $\underline{w}$ given by (3.4.1) could be approximated by optimal stopping value functions with truncated terminal cost. More explicitly, for any $n \in \mathbb{N}$, we define

$$\underline{w}^n(x) := \inf_{\tau\in\mathcal{T}_x} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau g(X_s)ds + G(X_\tau)\wedge n\right)\right], \quad x \in E. \tag{3.5.1}$$

Directly from the definition we get $\underline{w}^n(x) \leq \underline{w}^{n+1}(x) \leq \underline{w}(x)$ for any $x \in E$ and $n \in \mathbb{N}$. In Theorem 3.5.1 we show that in the limit we get $\underline{w}$; note that a similar property in the finite time horizon setting was shown in Step 3 of the proof of Theorem 3.3.2. The main difficulty here could be associated with the fact that, for any $x \in E$, the sequence $(\underline{w}^n(x))_{n\in\mathbb{N}}$ is increasing. Thus, in the proof we need to interchange the supremum with respect to n with the infimum with respect to τ.

Theorem 3.5.1. *For any $n \in \mathbb{N}$, let the maps $\underline{w}$ and $\underline{w}^n$ be given by (3.4.1) and (3.5.1), respectively. Then, we get*

$$\underline{w}(x) = \lim_{n\to\infty} \underline{w}^n(x), \quad x \in E.$$

Proof. Let us define the sequence of events $A_n := \{G(X_{\tau_n}) \leq n\}$, $n \in \mathbb{N}$, where

$$\tau_n := \inf\{t \geq 0 : \underline{w}^n(X_t) = G(X_t) \wedge n\}.$$

Noting that $G(\cdot) \wedge n \in \mathcal{C}_b^+(E)$ and using Theorem 3.4.11, we get that τ_n is an optimal stopping time for $\underline{w}^n(x)$, $n \in \mathbb{N}$, $x \in E$. Also, recalling that $g(\cdot) \geq 0$, for any $x \in E$ and $k \in \mathbb{N}$, we get

$$\begin{aligned} e^{G(x)} \geq e^{\underline{w}^k(x)} &= \mathbb{E}_x \left[e^{\int_0^{\tau_k} g(X_s)ds + G(X_{\tau_k})\wedge k}\right] \\ &\geq \mathbb{E}_x \left[1_{A_k^c} e^{\int_0^{\tau_k} g(X_s)ds + G(X_{\tau_k})\wedge k}\right] \geq \mathbb{P}_x\left[A_k^c\right] e^k. \end{aligned}$$

Thus $\mathbb{P}_x[A_k^c] \leq \frac{e^{G(x)}}{e^k}$ and $\sum_{k=1}^{\infty}\mathbb{P}_x[A_k^c] < \infty$, $x \in E$. Hence, from the Borel-Cantelli lemma, for any $x \in E$, we get

$$\mathbb{P}_x[\cup_{n=1}^{\infty}\cap_{k=n}^{\infty} A_k] = 1. \tag{3.5.2}$$

Let us fix $n \in \mathbb{N}$ and note that, on the event $\cap_{k=n}^{\infty}A_k$, for any $j \in \mathbb{N}$, we get

$$\begin{aligned}\underline{w}^{n+j+1}(X_{\tau_{n+j}}) \geq \underline{w}^{n+j}(X_{\tau_{n+j}}) &= G(X_{\tau_{n+j}}) \wedge (n+j)\\ &= G(X_{\tau_{n+j}}) \geq G(X_{\tau_{n+j}}) \wedge (n+j+1).\end{aligned}$$

Thus, noting that $\underline{w}^{n+j+1}(\cdot) \leq G(\cdot)\wedge(n+j+1)$, $j \in \mathbb{N}$, on the event $\cap_{k=n}^{\infty}A_k$, for any $j \in \mathbb{N}$, we get $\tau_{n+j+1} \leq \tau_{n+j}$. Combining this with (3.5.2), we may define the stopping time $\widetilde{\tau} := \lim_{n\to\infty}\tau_n$. Moreover, we get that $\widetilde{\tau} \in \mathcal{T}_x$, since, for any $n \in \mathbb{N}$, we get $\tau_n \in \mathcal{T}_x$. Thus, using the right-continuity of X and Fatou's lemma, for any $x \in E$, we get

$$\begin{aligned}e^{\underline{w}(x)} &\leq \mathbb{E}_x\left[e^{\int_0^{\widetilde{\tau}} g(X_s)ds + G(X_{\widetilde{\tau}})}\right]\\ &= \mathbb{E}_x\left[\lim_{n\to\infty} e^{\int_0^{\tau_n} g(X_s)ds + G(X_{\tau_n})\wedge n}\right]\\ &\leq \lim_{n\to\infty}\mathbb{E}_x\left[e^{\int_0^{\tau_n} g(X_s)ds + G(X_{\tau_n})\wedge n}\right] = \lim_{n\to\infty} e^{\underline{w}^n(x)} \leq e^{\underline{w}(x)}.\end{aligned}$$

Consequently, we get $\lim_{n\to\infty}\underline{w}^n(x) = \underline{w}(x)$, $x \in E$, which concludes the proof. $\square$

Remark 3.5.2. By analogy to Theorem 3.5.1, one could try to approximate the map $\overline{w}$ from (3.4.2) by the family

$$\overline{w}^n(x) := \inf_{\tau\in\mathcal{T}_{x,b}} \ln \mathbb{E}_x\left[e^{\int_0^{\tau} g(X_s)ds + G(X_\tau)\wedge n}\right],\ n \in \mathbb{N},\ x \in E.$$

However, since, for any $n \in \mathbb{N}$, the map $G(\cdot)\wedge n$ is bounded, using Theorem 3.4.11, we get $\underline{w}^n \equiv \overline{w}^n$ and, by Theorem 3.5.1, we get $\overline{w}^n \to \underline{w}$. $\Diamond$

3.5.2 Dyadic approximation

In this section, we investigate an approximation scheme related to the dyadic time-grid. We focus on the bounded case and throughout this section we assume $G \in \mathcal{C}_b^+(E)$. Hence, using Theorem 3.4.11, we get that the value functions from (3.4.1) and (3.4.2) are equal and, to simplify the notation, we set

$$\begin{aligned}w(x) &:= \inf_{\tau\in\mathcal{T}_x} \ln\mathbb{E}_x\left[\exp\left(\int_0^{\tau} g(X_s)ds + G(X_\tau)\right)\right],\\ &= \inf_{\tau\in\mathcal{T}_{x,b}} \ln\mathbb{E}_x\left[\exp\left(\int_0^{\tau} g(X_s)ds + G(X_\tau)\right)\right], \quad x \in E.\end{aligned} \tag{3.5.3}$$

In Theorem 3.5.3 we show that the map w from (3.5.3) may be approximated by a family optimal stopping problems on the dyadic time-grid, where the cost functions could be subject to some additional modifications. More explicitly, recall the families of stopping times $\mathcal{T}_x^m$ and $\mathcal{T}_{x,b}^m$ from Section 2.1. Let $(g_m)_{m\in\mathbb{N}}$ and $(G_m)_{m\in\mathbb{N}}$ be fixed sequences of functions from $\mathcal{C}_b^+(E)$. We assume that $g_m \uparrow g$ as $m \to \infty$ with $g_0(\cdot) \geq c_0 > 0$ and $G_m(x) \to G(x)$ as $m \to \infty$ uniformly in $x \in E$. Within this framework, we set

$$\begin{aligned} w^m(x) &:= \inf_{\tau\in\mathcal{T}_x^m} \ln \mathbb{E}_x\left[\exp\left(\int_0^\tau g_m(X_s)ds + G_m(X_\tau)\right)\right] \\ &= \inf_{\tau\in\mathcal{T}_{x,b}^m} \ln \mathbb{E}_x\left[\exp\left(\int_0^\tau g_m(X_s)ds + G_m(X_\tau)\right)\right], \quad m\in\mathbb{N},\, x\in E, \end{aligned} \tag{3.5.4}$$

where the second line follows from Theorem 3.2.15.

In Theorem 3.5.3 we show that $w^m(x)$ converges to $w(x)$ as $m \to \infty$ uniformly in x from compact sets. Note that this property provides a link between the continuous and discrete time optimal stopping frameworks. Indeed, for the constant sequences (g_m) and (G_m), the map w_m may be seen as a version of (3.5.3) with stopping times restricted to the dyadic time-grid. Thus, Theorem 3.5.3 implies that the continuous time stopping problem may be approximated by its dyadic counterpart even in the infinite time horizon; see Step 2 in the proof of Theorem 3.3.2 for a similar result in the finite time horizon. Also, this result helps to establish the existence of a solution to the continuous time optimality equation for impulse control problems; see Theorem 4.3.8 for details.

Theorem 3.5.3. *For any $m\in\mathbb{N}$, let the maps w and w^m be given by (3.5.3) and (3.5.4), respectively. Then, we get $w^m(x)\to w(x)$, as $m\to\infty$, uniformly in x from compact sets.*

Proof. We start with describing the structure of the proof. Recalling the non-negativity of g and G, we get $e^{w(x)} \geq 1$ and $e^{w^m(x)} \geq 1$, $m\in\mathbb{N}$, $x\in E$. Thus, using the inequality $|\ln y - \ln z| \leq \frac{1}{\min(y,z)}|y-z|$, $y,z>0$, we get

$$|w(x) - w^m(x)| \leq |W(x) - W^m(x)|, \quad x\in E,$$

where we set

$$\begin{aligned} W(x) &:= \inf_{\tau\in\mathcal{T}_x} \mathbb{E}_x\left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)}\right], \quad x\in E, \\ W^m(x) &:= \inf_{\tau\in\mathcal{T}_x^m} \mathbb{E}_x\left[e^{\int_0^\tau g_m(X_s)ds + G_m(X_\tau)}\right], \quad m\in\mathbb{N},\, x\in E. \end{aligned}$$

Consequently, it is enough to show that $W^m(x)$ converges to $W(x)$ as $m \to \infty$ uniformly in x from compact sets.

Next, define

$$W_m(x) := \inf_{\tau \in \mathcal{T}_x^m} \mathbb{E}_x \left[e^{\int_0^\tau g_m(X_s)ds + G(X_\tau)} \right], \quad m \in \mathbb{N},\, x \in E,$$

and note that

$$|W(x) - W^m(x)| \leq |W(x) - W_m(x)| + |W_m(x) - W^m(x)|, \quad m \in \mathbb{N},\, x \in E.$$

Thus, it is sufficient to show that $|W(x) - W_m(x)|$ and $|W_m(x) - W^m(x)|$ converge to zero uniformly in x from compact sets. Moreover, noting that $\mathcal{T}_x^m \subset \mathcal{T}_x$, $x \in E$, and $g_m \uparrow g$, we get

$$\underline{W}_m(x) \leq W_m(x) \leq \overline{W}_m(x), \quad m \in \mathbb{N},\, x \in E,$$

where the lower and upper bounds of W_m are given by

$$\underline{W}_m(x) := \inf_{\tau \in \mathcal{T}_x} \mathbb{E}_x \left[e^{\int_0^\tau g_m(X_s)ds + G(X_\tau)} \right], \quad m \in \mathbb{N},\, x \in E,$$
$$\overline{W}_m(x) := \inf_{\tau \in \mathcal{T}_x^m} \mathbb{E}_x \left[e^{\int_0^\tau g(X_s)ds + G(X_\tau)} \right], \quad m \in \mathbb{N},\, x \in E.$$

Hence, for the convergence of W_m to W, it is sufficient to show that both $\underline{W}_m$ and $\overline{W}_m$ converge to W uniformly on compact sets.

For transparency, we split the rest of the argument into three steps: (1) proof that $|\underline{W}_m - W| \to 0$; (2) proof that $|\overline{W}_m - W| \to 0$; (3) proof that $|W_m - W^m| \to 0$.

Step 1. We show that $|\underline{W}_m(x) - W(x)| \to 0$ as $m \to \infty$ uniformly in x from compact sets. Recalling that $g_m \uparrow g$ as $m \to \infty$, for any $x \in E$ and $m \in \mathbb{N}$, we get $W(x) \geq \underline{W}_m(x)$ and $\underline{W}_{m+1}(x) \geq \underline{W}_m(x)$. Thus, the limit $\underline{W}(x) := \lim_{m\to\infty} \underline{W}_m(x)$ is well defined, and $W(x) \geq \underline{W}(x)$, $x \in E$. Also, using Theorem 3.4.11, for any $m \in \mathbb{N}$ and $x \in E$, we get that the stopping time

$$\underline{\tau}_m := \inf \left\{ t \geq 0 : \underline{W}_m(X_t) = e^{G(X_t)} \right\}$$

is optimal for $\underline{W}_m(x)$. Since $\underline{\tau}_{m+1} \leq \underline{\tau}_m$, $m \in \mathbb{N}$, the limit $\underline{\tau} := \lim_{m\to\infty} \underline{\tau}_m$ is well defined and $\underline{\tau} \in \mathcal{T}_x$ as, for any $m \in \mathbb{N}$ and $x \in E$, we get $\underline{\tau}_m \in \mathcal{T}_x$. Then, using Fatou's lemma and the right-continuity of X, we get

$$\begin{aligned} W(x) &\geq \lim_{m\to\infty} \underline{W}_m(x) \\ &= \lim_{m\to\infty} \mathbb{E}_x \left[e^{\int_0^{\underline{\tau}_m} g_m(X_s)ds + G(X_{\underline{\tau}_m})} \right] \\ &\geq \mathbb{E}_x \left[\liminf_{m\to\infty} e^{\int_0^{\underline{\tau}_m} g_m(X_s) + G(X_{\underline{\tau}_m})} \right] = \mathbb{E}_x \left[e^{\int_0^{\underline{\tau}} g(X_s)ds + G(X_{\underline{\tau}})} \right] \geq W(x). \end{aligned}$$

Thus, we get $W(x) = \underline{W}(x)$, $x \in E$.

Using Theorem 3.4.11, we get that, for any $m \in \mathbb{N}$, the maps $x \mapsto \underline{W}_m(x)$ and $x \mapsto W(x)$ are continuous. Noting that, for any $x \in E$, the convergence $\underline{W}_m(x) \to W(x)$ as $m \to \infty$ is monotone, by Dini's theorem, we conclude that $\underline{W}_m(x) \to W(x)$ as $m \to \infty$ uniformly in x from compact sets.

Step 2. We show that $|\overline{W}_m(x) - W(x)| \to 0$ as $m \to \infty$ uniformly in x from compact sets. Noting that $\mathcal{T}_x^m \subset \mathcal{T}_x^{m+1} \subset \mathcal{T}_x$, we get $\overline{W}_{m+1}(x) \leq \overline{W}_m(x)$ and $W(x) \leq \overline{W}_m(x)$, $m \in \mathbb{N}$, $x \in E$. Thus, the limit $\overline{W}(x) := \lim_{m\to\infty} \overline{W}_m(x)$ is well defined and $W(x) \leq \overline{W}(x)$, $x \in E$.

Now, we show $\overline{W}(x) \leq W(x)$, $x \in E$. Note that, by Theorem 3.4.11, the stopping time $\widetilde{\tau} := \inf\{t \geq 0 : W(X_t) = e^{G(X_t)}\}$ is optimal for $W(x)$, $x \in E$. Next, let $\widetilde{\tau}_m$ denote $\mathcal{T}^m$ approximation of $\widetilde{\tau}$ given by

$$\widetilde{\tau}_m := \inf\{\tau \in \mathcal{T}^m \colon \tau \geq \widetilde{\tau}\} = \textstyle\sum_{j=1}^{\infty} 1_{\left\{\frac{j-1}{2^m} < \widetilde{\tau} \leq \frac{j}{2^m}\right\}} \frac{j}{2^m}.$$

Note that $\widetilde{\tau}_m \downarrow \widetilde{\tau}$, $m \to \infty$, and $\mathbb{E}_x\left[e^{\int_0^{\widetilde{\tau}} g(X_s)ds + G(X_{\widetilde{\tau}})}\right] = W(x) < \infty$, $x \in E$. Since $g(\cdot) \geq 0$ and $0 \leq G(\cdot) \leq \|G\|$, for any $m \in \mathbb{N}$, we get

$$e^{\int_0^{\widetilde{\tau}_m} g(X_s)ds + G(X_{\widetilde{\tau}_m})} \leq e^{\int_0^{\widetilde{\tau}+1} g(X_s)ds + \|G\|} \leq e^{\|g\|+\|G\|} e^{\int_0^{\widetilde{\tau}} g(X_s)ds + G(X_{\widetilde{\tau}})}.$$

Consequently, using Lebesgue's dominated convergence theorem, the fact that $\widetilde{\tau}_m \downarrow \widetilde{\tau}$, and the right-continuity of X, we get

$$\begin{aligned}
\overline{W}(x) = \lim_{m\to\infty} \overline{W}_m(x) &\leq \lim_{m\to\infty} \mathbb{E}_x\left[e^{\int_0^{\widetilde{\tau}_m} g(X_s)ds + G(X_{\widetilde{\tau}_m})}\right] \\
&= \mathbb{E}_x\left[\lim_{m\to\infty} e^{\int_0^{\widetilde{\tau}_m} g(X_s)ds + G(X_{\widetilde{\tau}_m})}\right] \\
&= \mathbb{E}_x\left[e^{\int_0^{\widetilde{\tau}} g(X_s)ds + G(X_{\widetilde{\tau}})}\right] = W(x),
\end{aligned}$$

which shows $\overline{W}(x) \leq W(x)$, for $x \in E$.

Using Theorem 3.2.15, we get that $x \mapsto \overline{W}_m(x)$ is continuous for any $m \in \mathbb{N}$. Also, using Theorem 3.4.11, we get that the map $x \mapsto W(x)$ is continuous, too. Recalling that the convergence $\overline{W}_m(x) \to W(x)$ is monotone, and using Dini's theorem, we conclude that $\overline{W}_m(x) \to W(x)$ as $m \to \infty$ uniformly in x from compact sets.

Step 3. We show that $|W_m(x) - W^m(x)| \to 0$ as $m \to \infty$ uniformly in $x \in E$. To simplify the notation, for any $m \in \mathbb{N}$ and $x \in E$, we set $a_m(x) := \max(W^m(x), W_m(x))$.

First, let $m \in \mathbb{N}$ and assume that $x \in E$ is such that $W^m(x) \leq W_m(x)$. Let τ^m be an optimal stopping time for W^m; using Theorem 3.2.15 we know that τ^m exists. Then, recalling that $G_m(\cdot) \geq 0$, we get

$$\begin{aligned}
0 \leq W_m(x) - W^m(x) &\leq \mathbb{E}_x \left[e^{\int_0^{\tau^m} g_m(X_s)ds} \left(e^{G(X_{\tau^m})} - e^{G_m(X_{\tau^m})} \right) \right] \\
&\leq \mathbb{E}_x \left[e^{\int_0^{\tau^m} g_m(X_s)ds + G_m(X_{\tau^m})} \right] \| e^G - e^{G_m} \| \\
&= W^m(x) \| e^G - e^{G_m} \| \leq a_m(x) \| e^G - e^{G_m} \|. \qquad (3.5.5)
\end{aligned}$$

Second, let us assume that $x \in E$ is such that $W^m(x) \geq W_m(x)$. Then, as in the previous case, we get

$$0 \leq W^m(x) - W_m(x) \leq a_m(x) \| e^G - e^{G_m} \|. \qquad (3.5.6)$$

Combining (3.5.5) and (3.5.6), for any $x \in E$ and $m \in \mathbb{N}$, we get

$$|W_m(x) - W^m(x)| \leq a_m(x) \| e^G - e^{G_m} \|.$$

Since $G_m \to G$ uniformly, for a sufficiently large m, we get

$$\| a_m \| \leq \max(\| W^m \|, \| W_m \|) \leq \max(e^{\| G_m \|}, e^{\| G \|}) \leq e^{\| G \| + 1}.$$

Combining this with the inequality $|e^z - e^y| \leq e^{\max(z,y)} |z - y|$, $z, y \in \mathbb{R}$, for a sufficiently large m, we get

$$|W_m(x) - W^m(x)| \leq e^{2\| G \| + 2} \| G_m - G \| \to 0, \quad x \in E,$$

which concludes the proof. □

Chapter 4

Impulse control problems

In this chapter, we consider risk-sensitive impulse control problems for continuous time Markov processes. We introduce suitable versions of optimality Bellman equations and construct optimal strategies for finite and infinite time horizon (long-run) problems. Our approach is based on probabilistic arguments and the results from Chapter 3.

This chapter is organised as follows: In Section 4.1, we state the main problem and introduce the set of assumptions. In Section 4.2, we discuss the finite time horizon impulse control problem. In particular, in Theorem 4.2.2 we provide a verification result, in Theorem 4.2.5 we construct a solution to the optimality equation, and in Proposition 4.2.9 we show a specific approximation result. Next, in Section 4.3, we discuss the long-run impulse control problem. In particular, in Theorem 4.3.4 we show a suitable verification argument and in Theorem 4.3.8 we show the existence of a solution to the associated Bellman equation. For the reader's convenience, in Section A.4 in Appendix A, we collect selected properties of the Multiplicative Poisson Equation, which are used in Section 4.3. Also, the assumptions introduced in Section 4.1 are illustrated by the examples presented in Section 5.2.

The results presented in this chapter are based mainly on Jelito et al. (2020), where the long-run impulse control problem were considered. However, note that in this chapter we provide a more detailed discussion on the finite time horizon stopping problems. In particular, we construct a generic optimal impulse control strategy for this class of problems while, in the paper, the analysis was restricted to the strategies with finitely many impulses; see Theorem 4.2.2 and Theorem 4.2.5 for details.

4.1 Problem statement and assumptions

In this section, we state the main problem and introduce the notation and assumptions used throughout this chapter. We focus on the long-run setting; necessary modifications for the finite time horizon case are discussed at the beginning of Section 4.2.

By $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ we denote a continuous time standard $\mathcal{C}_b$-Feller-Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. As in Section 2.2, with a starting point $x \in E$ and an impulse control strategy $V = (\tau_i, \xi_i)_{i=1}^{\infty} \in \mathbb{V}$, we associate the controlled process, the corresponding probability measure, and the expectation operator denoted by Y, $\mathbb{P}_{(x,V)}$, and $\mathbb{E}_{(x,V)}$, respectively. Recall that an impulse control strategy $V = (\tau_i, \xi_i)_{i=1}^{\infty}$ is described by sequences (τ_i) and (ξ_i) indicating impulse times and after-impulse states of the process, respectively, and $Y_{\tau_i^-}$ denotes the state of Y right before the ith impulse; see Section 2.2 for details. Throughout this chapter we assume that after-impulse states are restricted to some fixed compact set $U \subseteq E$, i.e. $\xi_i \in U$, $i \in \mathbb{N}_*$.

The main focus of this chapter is set on the minimising the long-run risk-sensitive impulse control problem

$$\inf_{V\in\mathbb{V}} \limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{E}_{(x,V)} \left[\exp\left(\int_0^T f(Y_s)ds + \sum_{i=1}^{\infty} 1_{\{\tau_i\leq T\}} c(Y_{\tau_i^-}, \xi_i)\right)\right], \tag{4.1.1}$$

where $f \in \mathcal{C}_b^+(E)$ and $c \in \mathcal{C}_b^+(E\times U)$ are a *running reward/cost function* and a *shift-cost function*, respectively. Under a suitable regularity conditions, we find the optimal value and an optimal strategy for (4.1.1). In fact, we prove the existence of a solution to the suitable Bellman equation and link it to the optimal strategy; see Theorem 4.3.4 and Theorem 4.3.8. The argument is based on the dyadic approximation of the problem and utilises some results from Pitera and Stettner (2021). More specifically, in that paper the local span contraction technique is used to prove the existence of a solution to the associated optimality equation in the dyadic framework; see Section A.3 for details. In the present chapter we combine these results with the regularity properties of the optimal stopping problems that were discussed in Chapter 3 and show how this can be used to deduce the existence of a solution to the suitable optimality equation in the continuous time case. To embed the associated optimal stopping problems into the framework considered in Chapter 3, we use the change of measure transformation based on a solution to the Multiplicative Poisson Equation; see Section A.4 for details.

Also, in this chapter we study a finite time horizon version of (4.1.1). We characterise the optimal strategy and provide an efficient approximation scheme for the value function. The properties of finite time horizon functionals could be seen as a problem of independent interest. Also, they are used in Theorem 4.3.9 to obtain some results in the long-run setting.

In this chapter we assume the following conditions:

($\mathcal{B}1$) (After-impulse states constraints.) After-impulse states take values in some fixed compact set $U \subset E$, i.e. for any strategy $V = (\tau_i, \xi_i)_{i=1}^{\infty} \in \mathbb{V}$ we have $\xi_i \in U$, $i \in \mathbb{N}_*$.

($\mathcal{B}2$) (Reward/cost function constraints.) We have $f \in \mathcal{C}_b^+(E)$.

($\mathcal{B}3$) (Impulse cost function constraints.) We have $c \in \mathcal{C}_b^+(E \times U)$. Also, c is bounded away from zero by $c_0 > 0$ and satisfies the triangle inequality, i.e. we have

$$c_0 \leq c(x,\xi) \leq c(x,\zeta) + c(\zeta,\xi), \quad x \in E,\, \xi,\zeta \in U. \tag{4.1.2}$$

Moreover, c satisfies *uniform limit at infinity* condition, i.e. we have

$$\lim_{\|x\|,\|y\|\to\infty} \sup_{\xi\in U} |c(x,\xi) - c(y,\xi)| = 0. \tag{4.1.3}$$

($\mathcal{B}4$) (Distance control.) For any compact set $\Gamma \subset E$, $t_0 > 0$, and $r_0 > 0$, we have

$$\lim_{r\to\infty} M_\Gamma(t_0, r) = 0, \qquad \lim_{t\to 0} M_\Gamma(t, r_0) = 0, \tag{4.1.4}$$

where $M_\Gamma(t,r) := \sup_{x\in\Gamma} \mathbb{P}_x[\sup_{s\in[0,t]} \rho(X_s, x) \geq r]$, $t, r > 0$.

($\mathcal{B}5$) (Mixing condition.) For any $t > 0$, we have $\mathcal{E}_t(x) = \mathcal{E}_t(y)$, $x, y \in E$, and

$$\sup_{x,y\in E} \sup_{A\in\mathcal{E}_t(y)} \frac{\mathbb{P}_x[X_t \in A]}{\mathbb{P}_y[X_t \in A]} < \infty, \tag{4.1.5}$$

where $\mathcal{E}_t(y) := \{A \in \mathcal{E} \colon \mathbb{P}_y[X_t \in A] > 0\}$. Also, for any $t > 0$ and $x \in E$, we have $U \in \mathcal{E}_t(x)$.

Let us now provide more extensive comments on these assumptions.

Assumption ($\mathcal{B}1$) is a standard condition stating that we may shift the process only to some compact subset of the state space. From the economic point of view, this assumption may reflect limited impact of a decision-maker on the market. Also, it helps to get the measurability of optimal strategies; see the discussion following (4.2.7) for details.

Assumption ($\mathcal{B}2$) constitutes classic reward/cost function constraints. Note that the non-negativity condition for f is only a technical normalisation. Indeed, for a generic $\widetilde{f} \in \mathcal{C}_b(E)$ we may add to the both sides of (4.1.1) the quantity $\|\widetilde{f}\|$ and set $f(\cdot) := \widetilde{f}(\cdot) + \|\widetilde{f}\| \in \mathcal{C}_b^+(E)$; see the similar comment for ($\mathcal{A}1$) in Section 3.1.

Assumption ($\mathcal{B}3$) imposes several restrictions on the cost function c. In particular, (4.1.2) is a standard assumption that can be used to exclude multiple impulses at the same time. More specifically, based on (4.1.2), we get that when one considers an impulse from x to ζ followed by an immediate impulse from ζ to ξ, it is rational to apply an impulse from x directly to ξ, without the intermediate point ζ. Next, the condition (4.1.3) could be used to show a uniform convergence of a suitable approximation of the optimality equation associated with (4.1.1); see Theorem 4.3.8 for details. In the following example we show that Assumption ($\mathcal{B}3$) is satisfied for many distance-like cost functions.

Example 4.1.1. Let $h \in \mathcal{C}_b^+(\mathbb{R}_+)$ be an increasing map satisfying subadditivity condition $h(x+y) \leq h(x) + h(y)$, $x, y \geq 0$. Also, let $c_0 > 0$ and recall that ρ is a metric on E. Then, the function $c(x,\xi) := h(\rho(x,\xi)) + c_0$, $x \in E$, $\xi \in U$, satisfies ($\mathcal{B}3$). For example, we may set $h_1(x) := x \wedge K$ with some $K \geq 0$, $h_2(x) = \frac{x}{1+x}$, or $h_3(x) := \frac{1}{1+e^{-x}}$.

To see that (4.1.2) is satisfied, it is enough to note that from the monotonicity and subadditivity conditions, for any $x \in E$ and $\xi, \zeta \in U$, we get

$$\begin{aligned} c(x,\xi) = h(\rho(x,\xi)) + c_0 &\leq h(\rho(x,\zeta) + \rho(\zeta,\xi)) + 2c_0 \\ &\leq h(\rho(x,\zeta)) + c_0 + h(\rho(\zeta,\xi)) + c_0 \\ &= c(x,\zeta) + c(\zeta,\xi). \end{aligned}$$

Also, to see (4.1.3), note that, from the compactness of U, we get $\rho(x,\xi) \to \infty$ as $\|x\| \to \infty$ uniformly in $\xi \in U$. Thus, recalling the monotonicity and the boundedness of h, we get $h(\rho(x,\xi)) \to \|h\|$ as $\|x\| \to \infty$ uniformly in $\xi \in U$. Consequently, we get

$$\lim_{\|x\|,\|y\| \to \infty} \sup_{\xi \in U} |h(\rho(x,\xi)) - h(\rho(y,\xi))| = 0,$$

which shows (4.1.3). ◆

Assumption ($\mathcal{B}4$) facilitates distance control of the uncontrolled process. In fact, this condition is identical with Assumption ($\mathcal{A}4$) that was used in Chapter 3.

Assumption ($\mathcal{B}5$) quantifies ergodic properties of the underlying uncontrolled process X. Equivalently, we can state that, for any $t > 0$, there exists a probability measure ν_t on $(E, \mathcal{E})$, a density $p_t : E \times E \to \mathbb{R}_+$, and constants $0 < a_t \leq b_t < \infty$, such that

$$\mathbb{P}_x[X_t \in A] = \int_A p_t(x, y)\nu_t(dy), \quad A \in \mathcal{E}, \tag{4.1.6}$$

with $\nu_t(U) > 0$ and $a_t \leq p_t(x, y) \leq b_t$, $x, y \in E$. Indeed, from ($\mathcal{B}5$), for any fixed $t > 0$, we get that the distributions $\mathbb{P}_x[X_t \in \cdot]$, $x \in E$, are equivalent. Thus, setting $\nu_t(\cdot) := \mathbb{P}_{x_0}[X_t \in \cdot]$, where $x_0 \in E$ is some fixed point, and noting that, from (4.1.5), the density of $\mathbb{P}_x[X_t \in \cdot]$ with respect to $\nu_t(\cdot)$ must be bounded, we get (4.1.6). Also, from (4.1.6) we directly get ($\mathcal{B}5$), thus these conditions are equivalent. Next, note that (4.1.6) is satisfied e.g. for regular reflected diffusions in bounded domains; see Section 5.2 for details. Also, it should be noted that this condition implies the global minorisation property as well as the existence of a solution to the Multiplicative Poisson Equation. Namely, under ($\mathcal{B}5$), we get the following two properties:

($\mathcal{B}5$a) (Global minorisation.) For any $t > 0$, there exists $a_t > 0$ and a probability measure ν_t on $(E, \mathcal{E})$, such that $\nu_t(U) > 0$ and

$$\inf_{x \in E} \mathbb{P}_x[X_t \in A] \geq a_t \nu_t(A), \quad A \in \mathcal{E}.$$

($\mathcal{B}5$b) (Existence of MPE solution.) There exists a map $v \in \mathcal{C}_b(E)$ satisfying the Multiplicative Poisson Equation

$$v(x) = \ln \mathbb{E}_x \left[\exp \left(\int_0^t (f(X_s) - r(f))ds + v(X_t) \right) \right], \quad x \in E,\, t \geq 0, \tag{4.1.7}$$

where $r(f)$ is the semi-group type given by

$$r(f) := \inf_{t > 0} \frac{1}{t} \ln \sup_{x \in E} \mathcal{P}_t^f \mathbb{1}(x) \tag{4.1.8}$$

with $\mathcal{P}_t^f$ given by (2.1.5) and $\mathbb{1}$ denoting the function identically equal to 1.

Condition ($\mathcal{B}5$a) is a direct consequence of (4.1.6) while ($\mathcal{B}5$b) follows from Theorem A.4.1 in Section A.4. In fact, all the proofs presented in this chapter are valid if we only assume the global minorisation and the existence of a solution to the Multiplicative Poisson Equation, i.e. replace ($\mathcal{B}5$) with ($\mathcal{B}5$a) and ($\mathcal{B}5$b). Nevertheless, we introduced stronger condition ($\mathcal{B}5$) to simplify the narrative.

Condition ($\mathcal{B}5$a) constitutes the global minorisation property of the uncontrolled process and could be linked to the (global) Doeblin condition and uniform ergodicity; see Hairer and Mattingly (2011) for details. Also, this condition is used in Pitera and Stettner (2021) to prove the existence of a solution to the dyadic version of (4.3.3); see Assumption (A.4) therein and Proposition A.3.1 in this thesis.

Condition ($\mathcal{B}5$b) gives the existence of a solution to the Multiplicative Poisson Equation. With this solution we may associate a change of measure transformation that simplifies some stochastic optimisation problems. We refer to Section A.4 for a more detailed discussion on the Multiplicative Poisson Equation. Also, more specific comments on the use of change of measure transformation associated with (4.1.7) can be found in Section 4.3; see Equation (4.3.5) and the following discussion.

4.2 Finite time horizon impulse control

In this section, we study the finite time horizon version of (4.1.1). More explicitly, for any $T \geq 0$, $x \in E$, and $V \in \mathbb{V}$, we define

$$J_T(x, V) := \ln \mathbb{E}_{(x,V)} \left[\exp \left(\int_0^T f(Y_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} c(Y_{\tau_i^-}, \xi_i) \right) \right]. \quad (4.2.1)$$

With the help of a suitable form of the Bellman equation, we construct an optimal impulse control strategy for

$$\inf_{V \in \mathbb{V}} J_T(x, V), \quad T \geq 0, \, x \in V. \quad (4.2.2)$$

Also, we show several approximation results, including limits of problems with finitely many impulses. These results, apart from their own merit, are used in the proof of Theorem 4.3.9. In this section we assume ($\mathcal{B}1$)–($\mathcal{B}4$); Assumption ($\mathcal{B}5$) is not needed here. Also, in Assumption ($\mathcal{B}3$) we may omit the uniform limit at infinity condition (4.1.3).

4.2.1 Verification theorem

To solve (4.2.2), we show the existence of a solution $u_T \in \mathcal{C}_b^+([0, T] \times E)$ to the associated Bellman equation given, for any $T \geq 0$, $t \in [0, T]$, and $x \in E$, by

$$u_T(t,x) = \inf_{\tau \le T-t} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau f(X_s)ds + 1_{\{\tau < T-t\}} M u_T(t+\tau, X_\tau) \right) \right], \tag{4.2.3}$$

where the operator $M\colon \mathcal{C}_b^+([0,\infty)\times E) \to \mathcal{C}_b^+([0,\infty)\times E)$ is defined as

$$Mh(t,x) := \inf_{\xi\in U} (c(x,\xi)+h(t,\xi)), \quad h\in \mathcal{C}_b^+([0,\infty)\times E),\ t\ge 0,\ x\in E. \tag{4.2.4}$$

In fact, we show that $u_T(0,x)$ gives the optimal value of $\inf_{V\in\mathbb{V}} J_T(x,V)$, $T\ge 0$, $x\in E$. Also, we show that optimal impulse times could be associated with optimal stopping times for u_T, while optimal after-impulse states are given by the minimisers for Mu_T; see Theorem 4.2.2 for details. Before we do this, let us characterise optimal stopping times associated with (4.2.3).

Proposition 4.2.1. *For any $T\ge 0$, let $u_T\in \mathcal{C}_b^+([0,T]\times E)$ be a solution to (4.2.3). Then, for any $T\ge 0$, $t\in[0,T]$, and $x\in E$, the stopping time*

$$\tau_T(t) := \inf\{s\ge 0 : u_T(t+s,X_s) = Mu_T(t+s,X_s)\} \wedge (T-t)$$

is optimal for $u_T(t,x)$. Also, for any $T\ge 0$, $t\in[0,T]$, and $x\in E$, the process

$$z_{T,t}(s) := \exp\left(\int_0^{s\wedge(T-t)} f(X_h)dh + u_T(t+s\wedge(T-t), X_{s\wedge(T-t)}) \right), \quad s\ge 0,$$

is a $\mathbb{P}_x$-submartingale and $(z_{T,t}(\tau_T(t)\wedge s)$, $s\ge 0$, is a $\mathbb{P}_x$-martingale.

Proof. We transform the optimal stopping problem associated with (4.2.3) into the framework of Theorem 3.3.4. Let us fix $T\ge 0$ and, for any $t\in[0,T]$ and $x\in E$, let us define $\widetilde{g}(t,x) := f(x)$ and $\widetilde{G}(t,x) := Mu_T(t,x)$. Also, for any $T'\in[0,T]$, $t'\in[0,T-T']$, we define

$$\underline{w}_{T'}(t',x) := \inf_{\tau\le T'} \ln \mathbb{E}_x \left[\exp\left(\int_0^\tau \widetilde{g}(t'+s,X_s)ds + 1_{\{\tau<T'\}} \widetilde{G}(t'+\tau, X_\tau) \right) \right].$$

Recalling Assumption ($\mathcal{B}2$), we get $\widetilde{g}\in \mathcal{C}_b^+([0,T]\times E)$. Also, recalling (4.2.4) and using the fact that $u_T\in \mathcal{C}_b^+([0,T]\times E)$, we get $\widetilde{G}\equiv Mu_T \in \mathcal{C}_b^+([0,T]\times E)$. Thus, using Theorem 3.3.4, we get that, for any $T'\in[0,T]$, $t'\in[0,T-T']$, and $x\in E$, the stopping time

$$\underline{\tau}_{T'}(t') := \inf\{s\ge 0\colon \underline{w}_{T'-s}(t'+s,X_s) = \widetilde{G}(t'+s,X_s)\} \wedge T' \tag{4.2.5}$$

is optimal for $\underline{w}_{T'}(t', x)$, the process

$$\underline{z}_{T',t'}(s) := e^{\int_0^{s\wedge T'} \widetilde{g}(t'+h, X_h)dh + \underline{w}_{T'-s\wedge T'}(t'+s\wedge T', X_{s\wedge T'})}, \quad s \geq 0,$$

is a $\mathbb{P}_x$-submartingale and $(\underline{z}_{T',t'}(\underline{\tau}_{T'}(t') \wedge s))$, $s \geq 0$, is a $\mathbb{P}_x$-martingale. Next, note that $u_T(t,x) = \underline{w}_{T-t}(t,x)$, $t \in [0,T]$, $x \in E$. Thus, for any $t \in [0,T]$ and $x \in E$, using (4.2.5) with $T' = T - t$ and $t' = t$, we get that the stopping time

$$\begin{aligned}\underline{\tau}_{T-t}(t) &= \inf\{s \geq 0 \colon \underline{w}_{T-t-s}(t+s, X_s) = \widetilde{G}(t+s, X_s)\} \wedge (T-t)\\ &= \inf\{s \geq 0 \colon u_T(t+s, X_s) = Mu_T(t+s, X_s)\} \wedge (T-t) = \tau_T(t)\end{aligned}$$

is optimal for $u_T(t,x)$. Using a similar argument, from the submartingale property of $(\underline{z}_{T',t'}(s))$, $s \geq 0$, and the martingale property of $(\underline{z}_{T',t'}(\underline{\tau}_{T'}(t') \wedge s))$, $s \geq 0$, we get that $(z_{T,t}(s))$, $s \geq 0$, is a $\mathbb{P}_x$-submartingale and $(z_{T,t}(\tau_T(t) \wedge s)$, $s \geq 0$, is a $\mathbb{P}_x$-martingale, which concludes the proof. □

Let us now provide a more detailed comment on the link between (4.2.3) and (4.2.2). Let $T \geq 0$ and $u_T \in \mathcal{C}_b^+([0,T] \times E)$ be a solution to (4.2.3); the existence of this map is shown in Theorem 4.2.5. We construct a strategy associated with u_T. First, for any $t \in [0,T]$ and $i = 1, 2, \ldots$, let us recursively define

$$\begin{aligned}\widetilde{\tau}_i(t) &:= \inf\{s \geq \widetilde{\tau}_{i-1}(t) : u_T(t+s, X_s^i) = Mu_T(t+s, X_s^i)\} \wedge (T-t),\\ \widetilde{\xi}_i(t) &:= \arg\min_{\xi \in U} \left(c(X^i_{\widetilde{\tau}_i(t)}, \xi) + u_T(t + \widetilde{\tau}_i(t), \xi) \right),\end{aligned} \tag{4.2.6}$$

where $\widetilde{\tau}_0(t) \equiv 0$. Next, we define the strategy $\widehat{V}(t) := (\widehat{\tau}_i(t), \widehat{\xi}_i(t))_{i=1}^{\infty} \in \mathbb{V}$, where, for any $i = 1, 2, \ldots$, we set

$$\begin{aligned}\widehat{\tau}_i(t) &:= \widetilde{\tau}_i(t) 1_{\{\widetilde{\tau}_i(t) < T-t\}} + \infty 1_{\{\widetilde{\tau}_i(t) = T-t\}}\\ \widehat{\xi}_i(t) &:= \widetilde{\xi}_i(t) 1_{\{\widetilde{\tau}_i(t) < T-t\}} + \xi_0 1_{\{\widetilde{\tau}_i(t) = T-t\}},\end{aligned} \tag{4.2.7}$$

and $\xi_0 \in U$ is some fixed point. Note that $(\widehat{\tau}_i(t))$ is a simple modification of $(\widetilde{\tau}_i(t))$ which excludes impulses at the terminal point $T - t$. Intuitively, the impulse times for the strategy $\widehat{V}(t)$ are given by optimal stopping times for (4.2.3) while the after-impulse states are given by the minimisers of Mu_T. Note that compactness of U and continuity of u_T guarantees the existence of a measurable minimiser.

The following theorem provides a verification result associated with the Bellman equation (4.2.3). In particular, it shows that the strategy $\widehat{V}(t)$ is optimal for (4.2.2).

Theorem 4.2.2. *Let $T\geq 0$ and assume that there exists $u_T\in\mathcal{C}_b^+([0,T]\times E)$, which is a solution to (4.2.3). Also, for any $t\in[0,T]$, let the strategy $\widehat{V}(t):=(\widehat{\tau}_i(t),\widehat{\xi}_i(t))_{i=1}^\infty\in\mathbb{V}$ be given by (4.2.7). Then, we get*

$$u_T(t,x)=\inf_{V\in\mathbb{V}}J_{T-t}(x,V)=J_{T-t}(x,\widehat{V}(t)),\quad t\in[0,T],\, x\in E. \tag{4.2.8}$$

Proof. Let us fix some $T\geq 0$, $t\in[0,T]$, and $x\in E$. For transparency, we split the proof into two steps: (1) proof of $u_T(t,x)=J_{T-t}(x,\widehat{V}(t))$; (2) proof of $u_T(t,x)\leq J_{T-t}(x,V)$, $V\in\mathbb{V}$. Note that these two properties directly imply (4.2.8). Also, before we proceed, let us mention that the reader may consult the proof of Theorem 4.3.4, where a similar argument is considered. In fact, in the proof of Theorem 4.3.4 we do not need to take care of the variable t, which simplifies some technical arguments.

Step 1. We show that $u_T(t,x)=J_{T-t}(x,\widehat{V}(t))$. In the following, for the notational convenience, we write $\widehat{V}$ instead of $\widehat{V}(t)$; similar convention is used for $\widetilde{\tau}_i(t)$, $\widetilde{\xi}_i(t)$, $\widehat{\tau}_i(t)$, $\widehat{\xi}_i(t)$, $i=1,2,\ldots$

Using Proposition 4.2.1, we get that $\widetilde{\tau}_1$ is an optimal stopping time for $u_T(t,x)$. Also, we get that the process

$$\exp\left(\int_0^{\widetilde{\tau}_1\wedge s}f(X_h^1)dh+u_T(t+\widetilde{\tau}_1\wedge s,X^1_{\widetilde{\tau}_1\wedge s})\right),\quad s\geq 0,$$

is a $\mathbb{P}_{(x,\widehat{V})}$-martingale. Hence, using the strong Markov property, we get

$$e^{u_T(t+\widetilde{\tau}_1,X^2_{\widetilde{\tau}_1})}=\mathbb{E}_{(x,\widehat{V})}\left[e^{\int_{\widetilde{\tau}_1}^{\widetilde{\tau}_2}f(X_s^2)ds+1_{\{\widetilde{\tau}_2<T-t\}}Mu_T(t+\widetilde{\tau}_2,X^2_{\widetilde{\tau}_2})}\middle|\widehat{\mathcal{F}}^1_{\tau_1}\right]\quad \mathbb{P}_{(x,\widehat{V})}\text{ a.s.}$$

Also, from (4.2.6), for any $n=1,2,\ldots$, on the event $\{\widetilde{\tau}_n<T-t\}$, we get

$$u_T(t+\widetilde{\tau}_n,X^n_{\widetilde{\tau}_n})=Mu_T(t+\widetilde{\tau}_n,X^n_{\widetilde{\tau}_n})=c(X^n_{\widetilde{\tau}_n},X^{n+1}_{\widetilde{\tau}_n})+u_T(t+\widetilde{\tau}_n,X^{n+1}_{\widetilde{\tau}_n}).$$

Thus, recursively we get

$$\begin{aligned}
&u_T(t,x)\\
&\quad=\ln\mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^{\widetilde{\tau}_1}f(Y_s)ds+1_{\{\widetilde{\tau}_1<T-t\}}Mu_T(t+\widetilde{\tau}_1,X^1_{\widetilde{\tau}_1})}\right]\\
&\quad=\ln\mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^{\widetilde{\tau}_1}f(Y_s)ds+1_{\{\widetilde{\tau}_1<T-t\}}c(X^1_{\widetilde{\tau}_1},X^2_{\widetilde{\tau}_1})+1_{\{\widetilde{\tau}_1<T-t\}}u_T(t+\widetilde{\tau}_1,X^2_{\widetilde{\tau}_1})}\right]\\
&\quad=\ln\mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^{\widetilde{\tau}_n}f(Y_s)ds+\sum_{i=1}^n 1_{\{\widetilde{\tau}_i<T-t\}}c(X^i_{\widetilde{\tau}_i},X^{i+1}_{\widetilde{\tau}_i})+1_{\{\widetilde{\tau}_n<T-t\}}u_T(t+\widetilde{\tau}_n,X^{n+1}_{\widetilde{\tau}_n})}\right].
\end{aligned} \tag{4.2.9}$$

Combining the boundedness of f and u_T with Fatou's lemma, we get

$$\ln \mathbb{E}_{(x,\widehat{V})}\left[e^{\sum_{i=1}^{\infty} 1_{\{\widetilde{\tau}_i<T-t\}} c(X^i_{\widetilde{\tau}_i},\widetilde{\xi}_i))}\right] \leq u_T(t,x)+T\|f\|+\|u_T\|<\infty.$$

Consequently, recalling that $c(\cdot,\text{-}) \geq c_0 > 0$, we get $\widetilde{\tau}_n \to T-t$ as $n \to \infty$. Hence, noting that $u_T(T,x)=0$, $x \in E$, and using the continuity of u_T, we get

$$u_T(t+\widetilde{\tau}_n, Y_{\widetilde{\tau}_n}) \to u_T(T, Y_{T-t}) = 0, \quad n \to \infty.$$

Thus, letting $n \to \infty$ in (4.2.9), using Lebesgue's dominated convergence theorem, and recalling (4.2.6) and (4.2.7), we get

$$\begin{aligned} u_T(t,x) &= \ln \mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^{T-t} f(Y_s)ds+\sum_{i=1}^{\infty} 1_{\{\widetilde{\tau}_i<T-t\}} c(X^i_{\widetilde{\tau}_i}, X^{i+1}_{\widetilde{\tau}_i})}\right] \\ &= \ln \mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^{T-t} f(Y_s)ds+\sum_{i=1}^{\infty} 1_{\{\widetilde{\tau}_i<T-t\}} c(X^i_{\widetilde{\tau}_i}, \widetilde{\xi}_i)}\right] \\ &= \ln \mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^{T-t} f(Y_s)ds+\sum_{i=1}^{\infty} 1_{\{\widehat{\tau}_i\leq T-t\}} c(X^i_{\widehat{\tau}_i}, \widehat{\xi}_i)}\right], \end{aligned}$$

which concludes the proof of $u_T(t,x) = J_{T-t}(x,\widehat{V})$.

Step 2. We show that $u_T(t,x) \leq J_{T-t}(x,V)$ for any strategy $V=(\tau_i,\xi_i) \in \mathbb{V}$. Noting that we act in the finite time horizon framework, without any loss of generality we assume that the impulse times τ_i take values in the set $[0,T-t) \cup \{\infty\}$; more general strategies do not improve the value of the problem. More specifically, for a generic $\bar{V}=(\bar{\tau}_i,\bar{\xi}_i) \in \mathbb{V}$ we may define the strategy $V := (\tau_i,\xi_i) \in \mathbb{V}$ with $\tau_i := \bar{\tau}_i 1_{\{\bar{\tau}_i<T-t\}} + \infty 1_{\{\bar{\tau}_i \geq T-t\}}$ and $\xi_i := \bar{\xi}_i 1_{\{\tau_i<T-t\}} + \xi_0 1_{\{\tau_i \geq T-t\}}$, $i=1,2,\ldots$, where $\xi_0 \in U$ is some fixed point. For these two strategies, we get $J_T(x,V) \leq J_T(x,\bar{V})$, $x \in E$. Also, since we consider the minimisation problem, we can restrict our attention to the strategies for which

$$\ln \mathbb{E}_{(x,V)}\left[e^{\sum_{i=1}^{\infty} 1_{\{\tau_i<T-t\}} c(X^i_{\tau_i}, X^{i+1}_{\tau_i}))}\right] < \infty. \tag{4.2.10}$$

Using Proposition 4.2.1, we get that the process

$$e^{\int_0^{s\wedge(T-t)} f(X^1_h)dh+u_T(t+s\wedge(T-t),X_{s\wedge(T-t)})}, \quad s \geq 0,$$

is a $\mathbb{P}_{(x,V)}$-submartingale. Hence, using the strong Markov property, on the event $\{\tau_1 < T-t\}$, we get

$$e^{u_T(t+\tau_1,X^2_{\tau_1})} \leq \mathbb{E}_{(x,V)}\left[e^{\int_{\tau_1\wedge(T-t)}^{\tau_2\wedge(T-t)} f(X^2_s)ds+1_{\{\tau_2<T-t\}} M u_T(t+\tau_2,X^2_{\tau_2})} \,\middle|\, \widehat{\mathcal{F}}^1_{\tau_1}\right] \mathbb{P}_{(x,V)} \text{ a.s.}$$

Also, from (4.2.3) and (4.2.4), for any $n = 1, 2, \ldots$, on the event $\{\tau_n < T - t\}$, we get

$$u_T(t + \tau_n, X^n_{\tau_n}) \le Mu_T(t + \tau_n, X^n_{\tau_n}) \le c(X^n_{\tau_n}, X^{n+1}_{\tau_n}) + u_T(t + \tau_n, X^{n+1}_{\tau_n}).$$

Thus, recursively we get

$$\begin{aligned}
&u_T(t,x)\\
&\le \ln \mathbb{E}_{(x,V)}\left[e^{\int_0^{\tau_1\wedge(T-t)} f(Y_s)ds + 1_{\{\tau_1<T-t\}} Mu_T(t+\tau_1, X^1_{\tau_1})}\right]\\
&\le \ln \mathbb{E}_{(x,V)}\left[e^{\int_0^{\tau_1\wedge(T-t)} f(Y_s)ds + 1_{\{\tau_1<T-t\}} c(X^1_{\tau_1}, X^2_{\tau_1}) + 1_{\{\tau_1<T-t\}} u_T(t+\tau_1, X^2_{\tau_1})}\right]\\
&\le \ln \mathbb{E}_{(x,V)}\left[e^{\int_0^{\tau_n\wedge(T-t)} f(Y_s)ds + \sum_{i=1}^{n} 1_{\{\tau_i<T-t\}} c(X^i_{\tau_i}, X^{i+1}_{\tau_i}) + 1_{\{\tau_n<T-t\}} u_T(t+\tau_n, X^{n+1}_{\tau_n})}\right].
\end{aligned}$$

Recalling (4.2.10) and the fact that $c(\cdot,\text{-}) \ge c_0 > 0$, we get $\tau_n \wedge (T-t) \to T-t$ $\mathbb{P}_{(x,V)}$ a.s. as $n \to \infty$. Thus, letting $n \to \infty$, as in Step 1, we get

$$u_T(t,x) \le \ln \mathbb{E}_{(x,V)}\left[e^{\int_0^{T-t} f(Y_s)ds + \sum_{i=1}^{\infty} 1_{\{\tau_i<T-t\}} c(X^i_{\tau_i}, X^{i+1}_{\tau_i})}\right] = J_{T-t}(x,V),$$

which concludes the proof. □

4.2.2 Existence of a solution to the Bellman equation

To show the existence of a solution to the Bellman equation (4.2.3), we use an iterative procedure. For any $T \ge 0$, we define the family $(u^n_T)_{n\in\mathbb{N}}$ of functions $u^n_T\colon [0,T] \times E \to \mathbb{R}$ given recursively, for $n \in \mathbb{N}$. $t \in [0,T]$, and $x \in E$, by

$$\begin{aligned}
u^0_T(t,x) &:= \ln \mathbb{E}_x\left[e^{\int_0^{T-t} f(X_s)ds}\right],\\
u^{n+1}_T(t,x) &:= \inf_{\tau \le T-t} \ln \mathbb{E}_x\left[e^{\int_0^{\tau} f(X_s)ds + 1_{\{\tau<T-t\}} Mu^n_T(t+\tau, X_\tau)}\right].
\end{aligned} \tag{4.2.11}$$

We show that $u^n_T(x)$ gives the optimal value of the impulse control problem with at most n impulses; see Proposition 4.2.4 for details. Before we do this, let us show some basic properties of the maps u^n_T.

Proposition 4.2.3. *For any $T \ge 0$ and $n \in \mathbb{N}$, let u^n_T be given by (4.2.11). Then:*

(1) *For any $T \geq 0$ and $n \in \mathbb{N}$, the map $(t,x) \mapsto u_T^n(t,x)$ is jointly continuous and bounded. Also, for any $T \geq 0$ and $n \in \mathbb{N}$, the stopping time*

$$\tau_T^{n+1}(t) := \inf\{s \geq 0 : u_T(t+s, X_s) = Mu_T^n(t+s, X_s)\} \wedge (T-t). \quad (4.2.12)$$

is optimal for $u_T^{n+1}(t,x)$.

(2) *For any $T \geq 0$, $n \in \mathbb{N}$, and $x \in E$, the map $t \mapsto u_T^n(t,x)$ is decreasing.*

Proof. For transparency, we prove the claims point by point.

Proof of (1). We proceed by induction. First, we prove that the map $(t,x) \mapsto u_T^0(t,x)$ is jointly continuous. Noting that

$$|\ln z - \ln y| \leq \frac{1}{\min(z,y)}|z-y|, \quad z,y>0,$$
$$|e^z - e^y| \leq e^{\max(z,y)}|z-y|, \quad z,y \in \mathbb{R},$$

and using the boundedness of f, for $t,s \in [0,T]$, $y \in E$, and $L := e^{2T\|f\|}\|f\|$, we get

$$|u_T^0(t,y) - u_T^0(s,y)| \leq e^{T\|f\|}\mathbb{E}_y\left|e^{\int_0^{T-t} f(X_u)du} - e^{\int_0^{T-s} f(X_u)du}\right| \leq L|t-s|.$$

Also, using the $\mathcal{C}_b$-Feller property of X and Proposition 2.1.8, we get that the map $x \mapsto u_T^0(t,x)$ is continuous for any fixed $t \in [0,T]$. Thus, for any sequence $((t_k,x_k))_{k\in\mathbb{N}}$ converging to $(t,x) \in [0,T] \times E$, we get

$$\begin{aligned}|u_T^0(t_k,x_k) - u_T^0(t,x)| &\leq |u_T^0(t_k,x_k) - u_T^0(t,x_k)| + |u_T^0(t,x_k) - u_T^0(t,x)| \\ &\leq L|t_k - t| + |u_T^0(t,x_k) - u_T^0(t,x)| \to 0, \quad k \to \infty,\end{aligned}$$

which shows the continuity of $(t,x) \mapsto u_T^0(t,x)$.

Second, we show the continuity of $(t,x) \mapsto u_T^{n+1}(t,x)$ and the optimality of $\tau_T^{n+1}(t)$ for $n \in \mathbb{N}$. Let us assume that, for some $n \in \mathbb{N}$, we get that $(t,x) \mapsto u_T^n(t,x)$ is continuous. Combining this with the compactness of U, we get the joint continuity of the map $(t,x) \mapsto Mu_T^n(t,x)$. Thus, using Theorem 3.3.4, we get joint the continuity of $(t,x) \mapsto u_T^{n+1}(t,x)$ and the optimality of $\tau_T^{n+1}(t)$ for any $t \in [0,T]$, which concludes the proof of this part.

Proof of (2). We proceed by induction. The monotonicity of $t \mapsto u_T^0(t,x)$ follows directly from the definition and the non-negativity of f. Let $x \in E$ and assume that, for some $n \in \mathbb{N}$, the map $t \mapsto u_T^n(t,x)$ is decreasing. Noting that $t \mapsto Mu_T^n(t,x)$ is also decreasing, for any $t, h > 0$, such that $t + h \le T$, we get

$$\begin{aligned} u_T^{n+1}(t+h,x) &= \inf_{\tau\in\mathcal{T}} \ln \mathbb{E}_x \left[e^{\int_0^{\tau\wedge(T-t-h)} f(X_s)ds + 1_{\{\tau<T-t-h\}} Mu_T^n(t+h+\tau, X_\tau)} \right] \\ &\le \inf_{\tau\in\mathcal{T}} \ln \mathbb{E}_x \left[e^{\int_0^{\tau\wedge(T-t)} f(X_s)ds + 1_{\{\tau<T-t\}} Mu_T^n(t+\tau, X_\tau)} \right] \\ &= u_T^{n+1}(t,x), \end{aligned} \tag{4.2.13}$$

which concludes the proof. □

In the next proposition, we link the maps u_T^n from (4.2.11) to the optimal values of impulse control problems with finitely many impulses. Also, we show the Lipschitz continuity of $t \mapsto u_T^n(t,x)$.

Proposition 4.2.4. *For any $T \ge 0$ and $n \in \mathbb{N}$, let u_T^n be given by (4.2.11). Then:*

(1) *For any $t \in [0,T]$ and $x \in E$, we get*

$$u_T^n(t,x) = \inf_{V\in\mathbb{V}_n} J_{T-t}(x,V).$$

(2) *For any $t, h \in [0,T]$ and $x \in E$, we get*

$$|u_T^n(t,x) - u_T^n(h,x)| \le \|f\| |t-h|.$$

Proof. The proof of (1) follows the logic of the proof of Theorem 4.2.2 and is omitted for brevity.

Now we prove (2). Without any loss of generality, we assume $h > t$. Let $\varepsilon > 0$ and $\widehat{V}_\varepsilon^h = (\widehat{\tau}_i, \widehat{\xi}_i)_{i=1}^\infty \in \mathbb{V}_n$ be an ε-optimal strategy for $\inf_{V\in\mathbb{V}_n} J_{T-h}(x,V)$. For any $i = 1, 2, \ldots$, let us define

$$\begin{aligned} \widetilde{\tau}_i &:= \widehat{\tau}_i 1_{\{\widehat{\tau}_i \le T-h\}} + \infty 1_{\{\widehat{\tau}_i > T-h\}}, \\ \widetilde{\xi}_i &:= \widehat{\xi}_i 1_{\{\widehat{\tau}_i \le T-h\}} + \xi_0 1_{\{\widehat{\tau}_i > T-h\}}, \end{aligned}$$

where $\xi_0 \in U$ is some fixed point. Then, we get $J_{T-h}(x, \widehat{V}_\varepsilon^h) = J_{T-h}(x, \widetilde{V}_\varepsilon^h)$, where the strategy $\widetilde{V}_\varepsilon^h \in \mathbb{V}_n$ is given by $\widetilde{V}_\varepsilon^h := (\widetilde{\tau}_i, \widetilde{\xi}_i)_{i=1}^\infty$. Note that $\widetilde{V}_\varepsilon^h$ is a simple modification of $\widehat{V}_\varepsilon^h$, which excludes impulses after the time $T - h$.

Next, recalling point (1) of this proposition and the non-negativity of f, we get

$$\begin{aligned}
0 &\leq u_T^n(t,x) - u_T^n(h,x) \\
&\leq J_{T-t}(x, \widetilde{V}_\varepsilon^h) - J_{T-h}(x, \widetilde{V}_\varepsilon^h) + \varepsilon \\
&= \ln \mathbb{E}_{(x,\widetilde{V}_\varepsilon^h)} \left[e^{\int_0^{T-t} f(Y_s)ds + \sum_{i=1}^\infty 1_{\{\widetilde{\tau}_i \leq T-t\}} c(Y_{\widetilde{\tau}_i^-}, \widetilde{\xi}_i)} \right] \\
&\quad - \ln \mathbb{E}_{(x,\widetilde{V}_\varepsilon^h)} \left[e^{\int_0^{T-h} f(Y_s)ds + \sum_{i=1}^\infty 1_{\{\widetilde{\tau}_i \leq T-h\}} c(Y_{\widetilde{\tau}_i^-}, \widetilde{\xi}_i)} \right] + \varepsilon \\
&\leq \ln \mathbb{E}_{(x,\widetilde{V}_\varepsilon^h)} \left[e^{\int_0^{T-h} f(Y_s)ds + \sum_{i=1}^\infty 1_{\{\widetilde{\tau}_i \leq T-h\}} c(Y_{\widetilde{\tau}_i^-}, \widetilde{\xi}_i)} \right] + (h-t)\|f\| \\
&\quad - \ln \mathbb{E}_{(x,\widetilde{V}_\varepsilon^h)} \left[e^{\int_0^{T-h} f(Y_s)ds + \sum_{i=1}^\infty 1_{\{\widetilde{\tau}_i \leq T-h\}} c(Y_{\widetilde{\tau}_i^-}, \widetilde{\xi}_i)} \right] + \varepsilon \\
&= (h-t)\|f\| + \varepsilon.
\end{aligned} \tag{4.2.14}$$

Recalling that $\varepsilon > 0$ was arbitrary, we conclude the proof. □

Using Proposition 4.2.4 and noting that $\mathbb{V}_n \subset \mathbb{V}_{n+1}$, $n \in \mathbb{N}$, for any $T \geq 0$, $n \in \mathbb{N}$, $t \in [0,T]$, and $x \in E$, we get $u_T^{n+1}(t,x) \leq u_T^n(t,x)$. Thus, the map

$$u_T(t,x) := \lim_{n\to\infty} u_T^n(t,x), \quad T \geq 0,\ t \in [0,T],\ x \in E \tag{4.2.15}$$

is well defined. In Theorem 4.2.5 we show that u_T is a solution to (4.2.3). Note that this, combined with Theorem 4.2.2, shows the existence of an optimal strategy for (4.2.2).

Theorem 4.2.5. *For any $T \geq 0$, let the map u_T be given by (4.2.15). Then, we get that $(t,x) \mapsto u_T(t,x)$ is jointly continuous, bounded, non-negative, and satisfies (4.2.3).*

Proof. Let us fix $T \geq 0$. For any $t \in [0,T]$ and $x \in E$, recalling that the map $n \mapsto u_T^{n+1}(t,x)$ is decreasing, we get

$$\begin{aligned}
Mu_T(t,x) = M \lim_{n\to\infty} u_T^n(t,x) &= M \inf_{n\in\mathbb{N}} u_T^n(t,x) \\
&= \inf_{n\in\mathbb{N}} Mu_T^n(t,x) = \lim_{n\to\infty} Mu_T^n(t,x).
\end{aligned}$$

Next, recalling Proposition 4.2.4 and noting that the cost of the *no impulse* strategy is bounded from above by $T\|f\|$, for any $n \in \mathbb{N}$, we get

$$\|Mu_T^n\| \leq \|c\| + \|u_T^n\| \leq \|c\| + T\|f\|. \tag{4.2.16}$$

Also, recalling the non-negativity of c and f, and using Proposition 4.2.4, we get $u_T^n(t,x) \geq 0$, $n \in \mathbb{N}$, $t \in [0,T]$, $x \in E$. Thus, we get $Mu_T(t,x) \geq 0$, $t \in [0,T]$, $x \in E$. Also, for any $t \in [0,T]$ and $x \in E$, recalling that the map $n \mapsto Mu_T^n(t,x)$ is decreasing and using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
u_T(t,x) &= \lim_{n\to\infty} u_T^n(t,x) \\
&= \inf_{\tau \leq T-t} \inf_{n\in\mathbb{N}} \ln \mathbb{E}_x \left[e^{\int_0^\tau f(X_s)ds + 1_{\{\tau<T-t\}} Mu_T^{n-1}(t+\tau, X_\tau)} \right] \\
&= \inf_{\tau \leq T-t} \ln \mathbb{E}_x \left[e^{\int_0^\tau f(X_s)ds + 1_{\{\tau<T-t\}} Mu_T(t+\tau, X_\tau)} \right]. \qquad (4.2.17)
\end{aligned}$$

In particular, we get that u_T is a solution to (4.2.3). Also, recalling Proposition 4.2.4, for any $t, s \geq 0$ and $\xi \in E$, we get

$$|u_T(t,\xi) - u_T(s,\xi)| = \lim_{n\to\infty} |u_T^n(t,\xi) - u_T^n(s,\xi)| \leq |t-s| \|f\|, \quad \xi \in U,$$

thus, for any $t, s \geq 0$ and $x, y \in E$, we get

$$\begin{aligned}
|Mu_T(t,x) - Mu_T(s,y)| &\leq \sup_{\xi\in U} |c(x,\xi) - c(y,\xi)| + \sup_{\xi \in U} |u_T(t,\xi) - u_T(s,\xi)| \\
&\leq \sup_{\xi\in U} |c(x,\xi) - c(y,\xi)| + |t-s|\|f\|.
\end{aligned}$$

Consequently, we get $Mu_T \in \mathcal{C}_b^+([0,T] \times E)$. Thus, using (4.2.17) and Theorem 3.3.4, we get $u_T \in \mathcal{C}_b^+([0,T] \times E)$, which concludes the proof. □

Based on the construction of the map u_T, we may link (4.2.2) to optimisation problems with finitely many impulses. More specifically, we get that $\inf_{V\in\mathbb{V}} J_T(x,V)$ could be approximated by the optimal value functions for the impulse control strategies from $\mathbb{V}_n$. This is summarised in the following corollary.

Corollary 4.2.6. *For any $T \geq 0$, let the map J_T be given by* (4.2.1)*. Then, for any $x \in E$, we get*

$$\inf_{V\in\mathbb{V}} J_T(x,V) = \lim_{n\to\infty} \inf_{V\in\mathbb{V}_n} J_T(x,V).$$

Proof. Recall the maps u_T^n and u_T given by (4.2.11) and (4.2.15), respectively. Combining Theorem 4.2.2, Theorem 4.2.5, and Proposition 4.2.4, for any $T \geq 0$ and $x \in E$, we get

$$\inf_{V\in\mathbb{V}} J_T(x,V) = u_T(0,x) = \lim_{n\to\infty} u_T^n(0,x) = \lim_{n\to\infty} \inf_{V\in\mathbb{V}_n} J_T(x,V),$$

which concludes the proof. □

4.2.3 Approximation of finite time horizon impulse control

In this section, we present approximation results related to the finite time horizon impulse control problems. First, we show that, for any $n \in \mathbb{N}$, we may approximate $\inf_{V\in\mathbb{V}_n} J_T(x,V)$, $T \geq 0$, $x \in E$, by the optimal values of the corresponding impulse control problems with dyadic strategies. To do this, by analogy to (4.2.11), for any $T \geq 0$ and $m \in \mathbb{N}$, we recursively define the family of maps $(u_T^{n,m})_{n\in\mathbb{N}}$ given, for any $n \in \mathbb{N}$, $t \in [0,T]$, and $x \in E$, by

$$\begin{aligned} u_T^{0,m}(t,x) &:= \ln \mathbb{E}_x \left[e^{\int_0^{T-t} f(X_s)ds} \right], \\ u_T^{n+1,m}(t,x) &:= \inf_{\substack{\tau \leq T-t \\ \tau\in\mathcal{T}^m}} \ln \mathbb{E}_x \left[e^{\int_0^{\tau} f(X_s)ds + 1_{\{\tau<T-t\}} M u_T^{n,m}(t+\tau, X_\tau)} \right]. \end{aligned} \tag{4.2.18}$$

In the following, we link $u_T^{n,m}$ to $\inf_{V\in\mathbb{V}_n^m} J_T(x,V)$. Also, we show $u_T^{n,m} \to u_T^n$ as $m \to \infty$.

We start with linking $u_T^{n,m}$ to the optimal value of the impulse control problem with at most n impulses and impulse times from $\mathcal{T}^m$. The proof of Proposition 4.2.7 follows the lines of the proof of Theorem 4.2.2 and is omitted for brevity.

Proposition 4.2.7. *For any $n,m \in \mathbb{N}$, and $T \geq 0$, let the map $u_T^{n,m}$ be given by (4.2.18). Then, we get*

$$u_T^{n,m}(t,x) = \inf_{V\in\mathbb{V}_n^m} J_{T-t}(x,V), \quad t\in[0,T],\, x\in E.$$

Now, we show that the impulse control problem with finitely many impulses may be approximated by the optimization problems on the dyadic time-grid.

Proposition 4.2.8. *Let $T \geq 0$ and $n \in \mathbb{N}$. Also, let the maps u_T^n and $u_T^{n,m}$, $m \in \mathbb{N}$, be given by (4.2.11) and (4.2.18), respectively. Then, we get $u_T^{n,m}(t,x) \to u_T^n(t,x)$ as $m \to \infty$ uniformly on compacts, i.e. uniformly in $(t,x) \in [0,T]\times\Gamma$, where $\Gamma \subset E$ is a compact set.*

Proof. Let us fix $T \geq 0$ and proceed by induction with respect to n. The claim for $n = 0$ is straightforward as $u_T^{n,m} \equiv u_T^n$ for any $m \in \mathbb{N}$. Let $n \in \mathbb{N}$ and assume that $u_T^{n,m}(t,x) \to u_T^n(t,x)$ as $m \to \infty$ uniformly in $(t,x) \in [0,T]\times\Gamma$ for any compact set $\Gamma \subset E$. We show that $u_T^{n+1,m}(t,x) \to u_T^{n+1}(t,x)$ as $m \to \infty$ uniformly in $(t,x) \in [0,T]\times\Gamma$ for any compact set $\Gamma \subset E$.

Let us fix some compact set $\Gamma \subset E$. Also, to simplify the notation, for any $m \in \mathbb{N}$, $t \in [0,T]$, and $x \in E$, we set

$$U_T^{n+1}(t,x) := \exp\left(u_T^{n+1}(t,x)\right), \quad U_T^{n+1,m}(t,x) := \exp\left(u_T^{n+1,m}(t,x)\right).$$

Recalling the non-negativity of f and c, for any $m \in \mathbb{N}$, $t \in [0,T]$, and $x \in E$, we get $\min\Big(U_T^{n+1}(t,x), U_T^{n+1,m}(t,x)\Big) \geq 1$. Thus, using the inequality

$$|\ln y - \ln z| \leq \frac{1}{\min(y,z)}|y-z|, \quad y,z > 0,$$

for any $m \in \mathbb{N}$, $t \in [0,T]$, and $x \in E$, we get

$$0 \leq u_T^{n+1,m}(t,x) - u_T^{n+1}(t,x) \leq U_T^{n+1,m}(t,x) - U_T^{n+1}(t,x). \tag{4.2.19}$$

Thus, it is enough to show that $U_T^{n+1,m}(t,x) \to U_T^{n+1}(t,x)$ as $m \to \infty$ uniformly in $(t,x) \in [0,T] \times \Gamma$. Before we do that, we introduce some auxiliary notation and results.

Consider any $t \in [0,T]$, $x \in \Gamma$, and $\varepsilon > 0$. Let $\widehat{\tau}^{(T,t,x)} \leq T - t$ be an optimal stopping time for $U_T^{n+1}(t,x)$; using Proposition 4.2.3 we know that $\widehat{\tau}^{(T,t,x)}$ exists. For any $m \in \mathbb{N}$, we define $\mathcal{T}^m$ approximation of $\widehat{\tau}^{(T,t,x)}$ by

$$\widehat{\tau}_m^{(T,t,x)} := \left(\sum_{j=1}^{\lfloor (T-t)2^m \rfloor} 1_{\left\{\frac{j-1}{2^m} < \widehat{\tau}^{(T,t,x)} \leq \frac{j}{2^m}\right\}} \frac{j}{2^m} \right) \wedge (T-t),$$

where $\lfloor a \rfloor := \sup\{k \in \mathbb{Z} \colon k \leq a\}$ denotes the integer part of $a \in \mathbb{R}$. Note that in the following, for brevity, we write $\widehat{\tau}$ and $\widehat{\tau}_m$ instead of $\widehat{\tau}^{(T,t,x)}$ and $\widehat{\tau}_m^{(T,t,x)}$. Also, for $m \in \mathbb{N}$, $s, s' \in [0,T]$, and $y, z \in E$, we set

$$\begin{aligned}
Z_T^n(\widehat{\tau}) &:= \exp\left(\textstyle\int_0^{\widehat{\tau}} f(X_s)ds + 1_{\{\widehat{\tau} < T-t\}} M u_T^n(t+\widehat{\tau}, X_{\widehat{\tau}})\right), \\
A_T^{n,m}(s,y) &:= |u_T^{n,m}(s,y) - u_T^n(s,y)|, \\
B_T^{n,m}(s,s',y,z) &:= \left|M u_T^{n,m}(s,y) - M u_T^n(s',z)\right|, \\
C_T^n(s,s',y,z) &:= \sup_{\xi \in U} |u_T^n(s,\xi) - u_T^n(s',\xi)| + \sup_{\xi \in U} |c(y,\xi) - c(z,\xi)|.
\end{aligned}$$

Note that, by Proposition 4.2.3, the function $(s,s',y,z) \mapsto C_T^n(s,s',y,z)$ is jointly continuous and bounded. Moreover, by the induction assumption, we get that $A_T^{n,m}(s,y) \to 0$ as $m \to \infty$ uniformly in $(s,y) \in [0,T] \times \widehat{\Gamma}$ where $\widehat{\Gamma} \subset E$ is a compact set. Also, for any $m \in \mathbb{N}$, $s, s' \in [0,T]$, and $y, z \in E$, we get

$$
\begin{aligned}
B_T^{n,m}(s,s',y,z) &\le \sup_{\xi\in U}|u_T^{n,m}(s,\xi)-u_T^n(s',\xi)| + \sup_{\xi\in U}|c(y,\xi)-c(z,\xi)|\\
&\le \sup_{\xi\in U}|u_T^{n,m}(s,\xi)-u_T^n(s,\xi)| + \sup_{\xi\in U}|u_T^n(s,\xi)-u_T^n(s',\xi)|\\
&\quad + \sup_{\xi\in U}|c(y,\xi)-c(z,\xi)|\\
&\le \sup_{s''\in[0,T]}\sup_{\xi\in U} A_T^{n,m}(s'',\xi) + C_T^n(s,s',y,z). \qquad (4.2.20)
\end{aligned}
$$

Next, using ($\mathcal{B}4$), we can find $R>0$ such that

$$
\sup_{x\in\Gamma}\mathbb{P}_x\left[\sup_{s\in[0,T]}\rho(X_s,x)>R\right]\le\varepsilon. \qquad (4.2.21)
$$

Let $B:=\{x\in E\colon \rho(x,\Gamma)\le R+1\}$. Using the induction assumption and the compactness of U, we may find $m_0\in\mathbb{N}$ such that, for any $m\ge m_0$, we get

$$
\sup_{s\in[0,T]}\sup_{y\in B\cup U} A_T^{n,m}(s,y)\le\varepsilon.
$$

In particular, recalling (4.2.20), for $m\ge m_0$, $s,s'\in[0,T]$, and $y,z\in E$, we get

$$
B_T^{n,m}(s,s',y,z)\le\varepsilon + C_T^n(s,s',y,z). \qquad (4.2.22)
$$

Also, recalling the identity $\delta_m=2^{-m}$, $m\in\mathbb{N}$, and noting that C_T^n is uniformly continuous on $[0,T]\times[0,T]\times B\times B$ and $C_T^n(s,s,y,y)=0$, $s\in[0,T]$, $y\in E$, we may find $r_1>0$ and $m_1\in\mathbb{N}$ such that, for any $m\ge m_1$, we get

$$
\sup_{\substack{s,s'\in[0,T]\\ |s-s'|\le\delta_m}}\ \sup_{\substack{y,z\in B\\ \rho(y,z)\le r_1}} C_T^n(s,s',y,z)\le\varepsilon. \qquad (4.2.23)
$$

Let $r:=\min(r_1,\frac{1}{2})$. Using ($\mathcal{B}4$), we may find $m_2\in\mathbb{N}$ such that, for any $m\ge m_2$, we get

$$
\sup_{x\in B}\mathbb{P}_x\left[\sup_{s\in[0,\delta_m]}\rho(X_s,x)\ge r\right]\le\varepsilon \quad\text{and}\quad \|f\|\delta_m\le\varepsilon. \qquad (4.2.24)
$$

Finally, it is useful to note that we get the inequalities $0\le u_T^k(t,x)\le T\|f\|$ and $0\le u_T^{k,m}(t,x)\le T\|f\|$, $k,m\in\mathbb{N}$, $t\in[0,T]$, $x\in E$; these follow from Proposition 4.2.4, Proposition 4.2.7, the non-negativity of f and c, and the fact that the cost of *no impulse* strategy is bounded from above by $T\|f\|$.

Now we return to the proof of $U_T^{n+1,m}(t,x) \to U_T^{n+1}(t,x)$ as $m \to \infty$ uniformly in $t \in [0,T]$ and $x \in \Gamma$. Recalling the boundedness of f, using the inequality $0 \le \widehat{\tau}_m - \widehat{\tau} \le \delta_m$, and the fact that $\{\widehat{\tau}_m < T-t\} \subset \{\widehat{\tau} < T-t\}$, for any $t \in [0,T]$ and $x \in \Gamma$, we get

$$\begin{aligned}
0 &\le U_T^{n+1,m}(t,x) - U_T^{n+1}(t,x) \\
&\le \mathbb{E}_x\left[e^{\int_0^{\widehat{\tau}_m} f(X_s)ds + 1_{\{\widehat{\tau}_m < T-t\}} M u_T^{n,m}(t+\widehat{\tau}_m, X_{\widehat{\tau}_m})}\right] - \mathbb{E}_x\left[Z_T^n(\widehat{\tau})\right] \\
&\le e^{\|f\|\delta_m} \mathbb{E}_x\left[Z_T^n(\widehat{\tau}) e^{B_T^{n,m}(t+\widehat{\tau}_m, t+\widehat{\tau}, X_{\widehat{\tau}_m}, X_{\widehat{\tau}})}\right] - \mathbb{E}_x\left[Z_T^n(\widehat{\tau})\right]. \qquad (4.2.25)
\end{aligned}$$

Noting that $Z_T^n(\widehat{\tau}) \le e^{\|c\|+2T\|f\|}$ and $\|B_T^{n,m}\| \le 2\|c\| + 2T\|f\|$, $m \in \mathbb{N}$, and using (4.2.21) and (4.2.22), for any $t \in [0,T]$, $x \in \Gamma$, and $m \ge \max(m_0, m_1, m_2)$, we get

$$\begin{aligned}
&\mathbb{E}_x\left[Z_T^n(\widehat{\tau}) e^{B_T^{n,m}(t+\widehat{\tau}_m, t+\widehat{\tau}, X_{\widehat{\tau}_m}, X_{\widehat{\tau}})}\right] \\
&\quad = \mathbb{E}_x\left[(1_{\{\rho(X_{\widehat{\tau}}, X_0) \le R\}} + 1_{\{\rho(X_{\widehat{\tau}}, X_0) > R\}}) Z_T^n(\widehat{\tau}) e^{B_T^{n,m}(t+\widehat{\tau}_m, t+\widehat{\tau}, X_{\widehat{\tau}_m}, X_{\widehat{\tau}})}\right] \\
&\qquad \le e^{\varepsilon} \mathbb{E}_x\left[1_{\{\rho(X_{\widehat{\tau}}, X_0) \le R\}} Z_T^n(\widehat{\tau}) e^{C_T^n(t+\widehat{\tau}_m, t+\widehat{\tau}, X_{\widehat{\tau}_m}, X_{\widehat{\tau}})}\right] + \varepsilon K_1, \qquad (4.2.26)
\end{aligned}$$

where $K_1 := e^{3\|c\|+4T\|f\|}$. Let $D := \{\sup_{s\in[0,\delta_m]} \rho(X_s, X_0) \le r\}$. Using (4.2.23) and (4.2.24), on the event $\{\rho(X_{\widehat{\tau}}, X_0) \le R\}$, we get

$$\mathbb{E}_{X_{\widehat{\tau}}}\left[1_D \exp\left(\sup_{|s-s'|\le\delta_m} \sup_{s''\in[0,\delta_m]} C^n(s,s',X_{s''},X_0)\right)\right] \le e^{\varepsilon},$$

$$\mathbb{E}_{X_{\widehat{\tau}}}\left[1_{D^c} \exp\left(\sup_{|s-s'|\le\delta_m} \sup_{s''\in[0,\delta_m]} C^n(s,s',X_{s''},X_0)\right)\right] \le \varepsilon K_2,$$

where $K_2 := e^{2\|c\|+2T\|f\|}$. Thus, using the strong Markov property, for any $t \in [0,T]$ and $x \in \Gamma$, we get

$$\begin{aligned}
&\mathbb{E}_x\left[1_{\{\rho(X_{\widehat{\tau}}, X_0) \le R\}} Z_T^n(\widehat{\tau}) e^{C_T^n(t+\widehat{\tau}_m, t+\widehat{\tau}, X_{\widehat{\tau}_m}, X_{\widehat{\tau}})}\right] \\
&\quad \le \mathbb{E}_x\left[1_{\{\rho(X_{\widehat{\tau}}, X_0) \le R\}} Z_T^n(\widehat{\tau}) \mathbb{E}_{X_{\widehat{\tau}}}\left[e^{\sup_{|s-s'|\le\delta_m} \sup_{s''\in[0,\delta_m]} C^n(s,s',X_{s''},X_0)}\right]\right] \\
&\quad \le \mathbb{E}_x\left[1_{\{\rho(X_{\widehat{\tau}}, X_0) \le R\}} Z_T^n(\widehat{\tau}) (e^{\varepsilon} + \varepsilon K_2)\right] \\
&\quad \le e^{\varepsilon} \mathbb{E}_x\left[Z_T^n(\widehat{\tau})\right] + \varepsilon K_3,
\end{aligned}$$

where $K_3 := K_2 e^{\|c\|+2T\|f\|}$. Hence, recalling (4.2.26), we get

$$\mathbb{E}_x\left[Z_T^n(\widehat{\tau}) e^{B_T^{n,m}(t+\widehat{\tau}_m, t+\widehat{\tau}, X_{\widehat{\tau}_m}, X_{\widehat{\tau}})}\right] \le e^{2\varepsilon} \mathbb{E}_x\left[Z_T^n(\widehat{\tau})\right] + \varepsilon e^{\varepsilon} K_3 + \varepsilon K_1. \qquad (4.2.27)$$

Recalling that $\|f\|\delta_m \leq \varepsilon$ and combining (4.2.27) with (4.2.25), for any $t \in [0,T]$ and $x \in \Gamma$, we get

$$\begin{aligned} 0 \leq U_T^{n+1,m}(t,x) - U_T^{n+1}(t,x) &\leq \left(e^{3\varepsilon} - 1\right) \mathbb{E}_x \left[Z_T^n(\widehat{\tau})\right] + \varepsilon e^{2\varepsilon} K_3 + \varepsilon e^{\varepsilon} K_1 \\ &\leq \left(e^{3\varepsilon} - 1\right) e^{2T\|f\|+\|c\|} + \varepsilon e^{2\varepsilon} K_3 + \varepsilon e^{\varepsilon} K_1. \end{aligned}$$

Noting that the upper bound is uniform in $t \in [0,T]$ and $x \in \Gamma$, and recalling that $\varepsilon > 0$ was arbitrary, we get $U_T^{n+1,m}(t,x) \to U_T^{n+1}(t,x)$ as $m \to \infty$ uniformly on $(t,x) \in [0,T] \times \Gamma$. Thus, recalling (4.2.19), we conclude the proof. □

Using Proposition 4.2.8, we may generalise Corollary 4.2.6 to the dyadic framework. More specifically, we get that the optimal value of (4.2.2) could be approximated using strategies from $\mathbb{V}_n^m$; see the following proposition for details.

Proposition 4.2.9. *For any $T \geq 0$, let the map J_T be given by (4.2.1). Then, for any $x \in E$, we get*

$$\lim_{n\to\infty} \lim_{m\to\infty} \inf_{V\in\mathbb{V}_n^m} J_T(x,V) = \inf_{V\in\mathbb{V}} J_T(x,V). \tag{4.2.28}$$

Proof. For any $n \in \mathbb{N}$, using Proposition 4.2.7, Proposition 4.2.8, and Proposition 4.2.4, we get

$$\lim_{m\to\infty} \inf_{V\in\mathbb{V}_n^m} J_T(x,V) = \lim_{m\to\infty} u_T^{n,m}(0,x) = u_T^n(0,x) = \inf_{V\in\mathbb{V}^n} J_T(x,V).$$

Letting $n \to \infty$ and recalling Corollary 4.2.6, we conclude the proof. □

4.3 Long-run impulse control

In this section, we consider the long-run risk-sensitive impulse control problem. Our primary goal is to characterise the optimal value and an optimal strategy for

$$\inf_{V\in\mathbb{V}} J(x,V), \quad x \in E, \tag{4.3.1}$$

where, for any $x \in E$ and $V \in \mathbb{V}$, the functional J is given by

$$J(x,V) := \limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{E}_{(x,V)} \left[e^{\int_0^T f(Y_s)ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} c(Y_{\tau_i^-}, \xi_i)} \right]. \tag{4.3.2}$$

Assuming ($\mathcal{B}1$)–($\mathcal{B}5$) we construct a solution to the associated optimality equation and solve the problem.

4.3.1 Verification theorem

To solve (4.3.1), we show the existence of a function $u \in \mathcal{C}_b(E)$ and a constant $\lambda \in \mathbb{R}$ that satisfy the impulse control Bellman equation

$$u(x) = \inf_{\tau \in \mathcal{T}_{x,b}} \ln \mathbb{E}_x \left[\exp \left(\int_0^\tau (f(X_s) - \lambda) ds + Mu(X_\tau) \right) \right], \quad x \in E, \tag{4.3.3}$$

where, with a slight abuse of notation, the operator $M \colon \mathcal{C}_b(E) \to \mathcal{C}_b(E)$ is defined as

$$Mh(x) := \inf_{\xi \in U} \left(c(x, \xi) + h(\xi) \right), \quad h \in \mathcal{C}_b(E),\, x \in E. \tag{4.3.4}$$

With the help of a solution to (4.3.3) we construct an optimal strategy and find the optimal value of (4.3.1). More specifically, we show that, regardless of the initial point $x \in E$, we get $\lambda = \inf_{V \in \mathbb{V}} J(x, V)$. Also, we show that optimal impulse times for (4.3.1) are linked to optimal stopping times for u and optimal after-shift states are given by the minimisers for the operator M; see Theorem 4.3.4 for details.

Before we proceed, we show a simple lemma stating an upper bound for the constant λ from (4.3.3). Namely, we show that $\lambda \leq r(f)$, where $r(f)$ denotes the type of the semigroup given by (4.1.8).

Lemma 4.3.1. *Let $u \in \mathcal{C}_b(E)$ and $\lambda \in \mathbb{R}$ be a solution to (4.3.3). Then, we get $\lambda \leq r(f)$, where $r(f)$ is given by (4.1.8).*

Proof. Using the continuity of u and the compactness of U, we get $\|Mu\| < \infty$. Thus, for any $x \in E$ and $t > 0$, we get

$$\begin{aligned} u(x) &\leq \ln \mathbb{E}_x \left[e^{\int_0^t (f(X_s) - \lambda) ds + Mu(X_t)} \right] \\ &\leq \ln \sup_{y \in E} \mathbb{E}_y \left[e^{\int_0^t (f(X_s) - \lambda) ds} \right] + \|Mu\|. \end{aligned}$$

Hence, dividing by t, letting $t \to \infty$, and using (A.4.2), we get $0 \leq r(f - \lambda)$ and consequently $\lambda \leq r(f)$. $\square$

Next, we focus on the optimal stopping problem linked to (4.3.3). Note that, in principle, the term $f(\cdot) - \lambda$ could be non-positive, and the results from Chapter 3 cannot be directly applied. Thus, to embed (4.3.3) into the framework of Chapter 3, we use a change of measure transformation associated with a solution to the Multiplicative Poisson Equation. More specifically, with a solution to (4.1.7) we associate a family of probability measures $(\mathbb{Q}_x)$, $x \in E$, given by the Radon-Nikodym derivative

$$d\mathbb{Q}_x\big|_{\mathcal{F}_t} := Y_t(x) d\mathbb{P}_x\big|_{\mathcal{F}_t}, \quad x \in E,\, t \geq 0,$$

where $Y_t(x) := e^{-v(x)} e^{\int_0^t (f(X_s) - r(f))ds + v(X_t)}$, $t \geq 0$, $x \in E$. In Section A.4, we show that the (uncontrolled) process X is Markov under $(\mathbb{Q}_x)$; see Theorem A.4.6. Also, using Proposition A.4.4, for any $x \in E$, $\lambda \in \mathbb{R}$, $G \in \mathcal{C}_b(E)$, and $\tau \in \mathcal{T}_{x,b}$, we get

$$\mathbb{E}_x \left[e^{\int_0^\tau (f(X_s) - \lambda)ds + G(X_\tau)} \right] = e^{v(x)}\, \mathbb{E}_x^{\mathbb{Q}} \left[e^{(r(f) - \lambda)\tau + G(X_\tau) - v(X_\tau)} \right]. \tag{4.3.5}$$

Based on this identity, we may transform the optimal stopping problem related to (4.3.3) into the framework considered in Chapter 3. Namely, while $(f(\cdot) - \lambda)$ on the left-hand side of (4.3.5) could be sign-changing, using Lemma 4.3.1, we get that the term $r(f) - \lambda$ at the right-hand side is non-negative. In fact, in many cases, this constant is positive, and the assumptions from Chapter 3 are satisfied.

Change of measure transformation associated with (4.1.7) is used in Proposition 4.3.2 below, where we introduce a martingale associated with a solution to (4.3.3). Note that in the proposition we assume that $\lambda < r(f)$. The remaining degenerate case $\lambda = r(f)$ needs a special argument; see Theorem 4.3.9 for details.

Proposition 4.3.2. *Let $u \in \mathcal{C}_b(E)$ and $\lambda < r(f)$ be a solution to (4.3.3). Also, let*

$$\hat{\tau} := \inf\{t \geq 0 : u(X_t) = Mu(X_t)\}. \tag{4.3.6}$$

Then, for any $x \in E$, the process $z_u(t) := e^{\int_0^t (f(X_s) - \lambda)ds + u(X_t)}$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale and the process $(z_u(\hat{\tau} \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale.

Proof. The proof is based on the change of measure technique related to the Multiplicative Poisson Equation; see Section A.4 for details. Recalling v from ($\mathcal{B}$5b), we set

$$\begin{aligned} w(x) &:= u(x) - v(x) + \|Mu\| + \|v\|, \quad x \in E, \\ G(x) &:= Mu(x) - v(x) + \|Mu\| + \|v\|, \quad x \in E, \\ d &:= r(f) - \lambda. \end{aligned}$$

Thus, recalling the family of measures $(\mathbb{Q}_x)$ from (A.4.7), and using (4.3.3) and (A.4.8), we get

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}_{x,b}} \mathbb{E}_x^{\mathbb{Q}} \left[e^{d\tau + G(X_\tau)} \right], \quad x \in E. \tag{4.3.7}$$

Also, recalling (A.4.7), note that if for some $x \in E$ and $T \geq 0$, we get $\mathbb{P}_x[\tau \leq T] = 0$, then we also get

$$\mathbb{Q}_x[\tau \leq T] \leq e^{T\|f-r(f)\|+2\|v\|}\mathbb{P}_x[\tau \leq T] = 0. \tag{4.3.8}$$

Thus, denoting by $\mathcal{T}_{x,b}^{\mathbb{Q}}$ the family of $\mathbb{Q}_x$ a.s. bounded stopping times, we get $\mathcal{T}_{x,b} = \mathcal{T}_{x,b}^{\mathbb{Q}}$, $x \in E$, and consequently (4.3.7) could be rewritten as

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}_{x,b}^{\mathbb{Q}}} \mathbb{E}_x^{\mathbb{Q}}\left[e^{d\tau + G(X_\tau)}\right], \quad x \in E. \tag{4.3.9}$$

Also, noting that $d > 0$ and $G \in \mathcal{C}_b^+(E)$, and using Theorem A.4.6, we get that Assumptions ($\mathcal{A}1$)–($\mathcal{A}4$) from Section 3.1 are satisfied for the process $((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$. Thus, using Theorem 3.4.11, in (4.3.9) we may replace the family of $\mathbb{Q}_x$ a.s. bounded stopping times $\mathcal{T}_{x,b}^{\mathbb{Q}}$ with the family of $\mathbb{Q}_x$ a.s. finite stopping times $\mathcal{T}_x^{\mathbb{Q}}$, i.e. we get

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}_x^{\mathbb{Q}}} \mathbb{E}_x^{\mathbb{Q}}\left[e^{d\tau + G(X_\tau)}\right], \quad x \in E.$$

Also, we get that the stopping time

$$\hat{\tau} := \inf\{t \geq 0 : w(X_t) = G(X_t)\} \in \mathcal{T}_x^{\mathbb{Q}}$$

is optimal for w. Moreover, recalling the definitions of w and G, we get

$$\hat{\tau} = \inf\{t \geq 0 : u(X_t) = Mu(X_t)\}.$$

Next, let us define $z_w(t) := e^{td+w(X_t)}$, $t \geq 0$. Using Theorem 3.4.11, we get that, for any $x \in E$, the process $(z_w(t))$, $t \geq 0$, is a $\mathbb{Q}_x$-submartingale and the process $(z_w(\hat{\tau} \wedge t))$, $t \geq 0$, is a $\mathbb{Q}_x$-martingale. Thus, using Proposition A.4.4, we get that $(z_w(t))$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale and $(z_w(\hat{\tau} \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale, which concludes the proof. □

Remark 4.3.3. Based on Proposition 4.3.2 one may ask if $\hat{\tau}$ is an optimal stopping time for $u(x)$, $x \in E$. In particular, we would need to show that $\hat{\tau} \in \mathcal{T}_{x,b}$. From Theorem 3.4.11, we get that $\mathbb{Q}_x[\hat{\tau} < \infty] = 1$, $x \in E$. However, the measures $\mathbb{Q}_x$ and $\mathbb{P}_x$ need not to be equivalent and it is not clear if we even have $\hat{\tau} \in \mathcal{T}_x$. Consequently, $\hat{\tau}$ may not be an optimal stopping time for u. Still, note that the process $(z_u(\hat{\tau} \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale and this is one of the main building blocks of the construction of the impulse control strategy associated with (4.3.3); see Theorem 4.3.4 for details. ◊

Now we state the verification theorem linking a solution to (4.3.3) with an optimal strategy for (4.3.1). Note that the existence of a solution to (4.3.3) is proved in Theorem 4.3.8.

Let us fix $u \in \mathcal{C}_b(E)$ and $\lambda \in \mathbb{R}$ solving (4.3.3). Let $\widehat{V} := (\widehat{\tau}_i, \widehat{\xi}_i)_{i=1}^{\infty}$ be a strategy given recursively by

$$\begin{cases} \widehat{\tau}_i := \inf\{t \geq \widehat{\tau}_{i-1} : u(X_t^i) = Mu(X_t^i)\}, \\ \widehat{\xi}_i := \operatorname*{arg\,min}_{\xi \in U} \left[c(X_{\widehat{\tau}_i}^i, \xi) + u(\xi)\right] 1_{\{\widehat{\tau}_i < \infty\}} + \xi_0 1_{\{\widehat{\tau}_i = \infty\}}, \end{cases} \tag{4.3.10}$$

for $i = 1, 2, \ldots$, where $\widehat{\tau}_0 := 0$ and $\xi_0 \in U$ is some fixed point. An intuitive interpretation of the strategy $\widehat{V}$ is as follows: every time the controlled process enters the set $\{y \in E : u(y) = Mu(y)\}$, we apply an impulse and shift the process to a minimiser of $Mu(y)$. In Theorem 4.3.4 we show that $\widehat{V}$ is an optimal strategy for (4.3.1).

Theorem 4.3.4. *Let $u \in \mathcal{C}_b(E)$ and $\lambda < r(f)$ be a solution to (4.3.3). Also, let the strategy $\widehat{V}$ be given by (4.3.10). Then, we get*

$$\lambda = \inf_{V \in \mathbb{V}} J(x, V) = J(x, \widehat{V}), \quad x \in E. \tag{4.3.11}$$

Proof. Let us fix $x \in E$. As in Theorem 4.2.2, we split the proof into two steps: (1) proof of $\lambda = J(x, \widehat{V})$; (2) proof of $\lambda \leq J(x, V)$, $V \in \mathbb{V}$. Note that these two properties directly imply (4.3.11).

Step 1. We show that $\lambda = J(x, \widehat{V})$. Using Proposition 4.3.2 and a suitable embedding, we get that the process $e^{\int_0^{\widehat{\tau}_1 \wedge T}(f(X_s^1) - \lambda)ds + u(X_{\widehat{\tau}_1 \wedge T}^1)}$, $T \geq 0$, is a $\mathbb{P}_{(x, \widehat{V})}$-martingale; see the discussion in Section 2.2 for details. Also, note that, on the event $\{\widehat{\tau}_1 < \infty\}$, we get $\widehat{\xi}_1 = X_{\widehat{\tau}_1}^2$ and

$$u(X_{\widehat{\tau}_1}^1) = Mu(X_{\widehat{\tau}_1}^1) = c(X_{\widehat{\tau}_1}^1, X_{\widehat{\tau}_1}^2) + u(X_{\widehat{\tau}_1}^2).$$

Thus, for any $T \geq 0$, we get

$$\begin{aligned} e^{u(x)} &= \mathbb{E}_{(x, \widehat{V})}\left[e^{\int_0^{\widehat{\tau}_1 \wedge T}(f(Y_s) - \lambda)ds + u(X_{\widehat{\tau}_1 \wedge T}^1)}\right] \\ &= \mathbb{E}_{(x, \widehat{V})}\left[e^{\int_0^{\widehat{\tau}_1 \wedge T}(f(Y_s) - \lambda)ds + 1_{\{\widehat{\tau}_1 \leq T\}} c(X_{\widehat{\tau}_1}^1, X_{\widehat{\tau}_1}^2) + 1_{\{\widehat{\tau}_1 > T\}} u(X_T^1)} \times \right. \\ &\qquad \left. \times\, e^{1_{\{\widehat{\tau}_1 \leq T\}} u(X_{\widehat{\tau}_1}^2)}\right]. \end{aligned} \tag{4.3.12}$$

From the martingale property, for any $T \geq 0$, on $\{\widehat{\tau}_1 \leq T\}$, we get

$$e^{u(X^2_{\widehat{\tau}_1})} = \mathbb{E}_{(x,\widehat{V})}\left[e^{\int_{\widehat{\tau}_1\wedge T}^{\widehat{\tau}_2\wedge T}(f(X^2_s)-\lambda)ds + u(X^2_{\widehat{\tau}_2\wedge T})} \middle| \widehat{\mathcal{F}}^1_{\widehat{\tau}_1\wedge T}\right] \quad \mathbb{P}_{(x,\widehat{V})} \text{ a.s.}$$

Combining this identity with (4.3.12), we get

$$\begin{aligned} e^{u(x)} = \mathbb{E}_{(x,\widehat{V})} \Big[& e^{\int_0^{\widehat{\tau}_2\wedge T}(f(Y_s)-\lambda)ds + \sum_{i=1}^2 1_{\{\widehat{\tau}_i \leq T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i})} \times \\ & \times e^{1_{\{\widehat{\tau}_1 > T\}} u(X^1_T) + 1_{\{\widehat{\tau}_1 \leq T < \widehat{\tau}_2\}} u(X^2_T) + 1_{\{\widehat{\tau}_2 \leq T\}} u(X^2_{\widehat{\tau}_2})} \Big]. \end{aligned}$$

Thus, acting recursively, we get

$$\begin{aligned} e^{u(x)} = \mathbb{E}_{(x,\widehat{V})} \Big[& e^{\int_0^{\widehat{\tau}_n\wedge T}(f(Y_s)-\lambda)ds + \sum_{i=1}^n 1_{\{\widehat{\tau}_i \leq T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i})} \times \\ & \times e^{\sum_{i=0}^{n-1} 1_{\{\widehat{\tau}_i \leq T < \widehat{\tau}_{i+1}\}} u(X^{i+1}_T) + 1_{\{\widehat{\tau}_n \leq T\}} u(X^n_{\widehat{\tau}_n})} \Big]. \end{aligned} \tag{4.3.13}$$

Using Fatou's lemma and the boundedness of f and u, we get

$$\begin{aligned} e^{u(x)} &\geq \mathbb{E}_{(x,\widehat{V})} \Big[\liminf_{n\to\infty} e^{\int_0^{\widehat{\tau}_n\wedge T}(f(Y_s)-\lambda)ds + \sum_{i=1}^n 1_{\{\widehat{\tau}_i \leq T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i})} \times \\ & \qquad \times e^{\sum_{i=0}^{n-1} 1_{\{\widehat{\tau}_i \leq T < \widehat{\tau}_{i+1}\}} u(X^{i+1}_T) + 1_{\{\widehat{\tau}_n \leq T\}} u(X^n_{\widehat{\tau}_n})} \Big] \\ &\geq \mathbb{E}_{(x,\widehat{V})} \left[e^{T(-\|f\|-|\lambda|) + \sum_{i=1}^\infty 1_{\{\widehat{\tau}_i \leq T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i}) - \|u\|} \right]. \end{aligned}$$

Thus, noting that $e^{u(x)} < \infty$, we get

$$\mathbb{E}_{(x,\widehat{V})} \left[e^{\sum_{i=1}^\infty 1_{\{\widehat{\tau}_i \leq T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i})} \right] < \infty,$$

for any $T \geq 0$. Thus, recalling that, by ($\mathcal{B}3$), we get $c(\cdot,\text{-}) \geq c_0 > 0$, we conclude that $\widehat{\tau}_n \uparrow \infty$. Consequently, letting $n \to \infty$ in (4.3.13) and using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} e^{u(x)} = \mathbb{E}_{(x,\widehat{V})} \Big[& e^{\int_0^{T}(f(Y_s)-\lambda)ds + \sum_{i=1}^\infty 1_{\{\widehat{\tau}_i \leq T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i})} \times \\ & \times e^{\sum_{i=0}^{\infty} 1_{\{\widehat{\tau}_i \leq T < \widehat{\tau}_{i+1}\}} u(X^{i+1}_T)} \Big] \end{aligned} \tag{4.3.14}$$

Taking the logarithm of both sides, dividing by T, and recalling the boundedness of u, we get

$$\begin{aligned}\lambda &= \limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^T f(Y_s)ds + \sum_{i=1}^{\infty} 1_{\{\widehat{\tau}_i \le T\}} c(X^i_{\widehat{\tau}_i}, X^{i+1}_{\widehat{\tau}_i})}\right]\\ &= \limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{E}_{(x,\widehat{V})}\left[e^{\int_0^T f(Y_s)ds + \sum_{i=1}^{\infty} 1_{\{\widehat{\tau}_i \le T\}} c(Y_{\widehat{\tau}_i^-}, \widehat{\xi}_i)}\right],\end{aligned}$$

which concludes the proof of $\lambda = J(x, \widehat{V})$, $x \in E$.

Step 2. We show that, for any strategy $V = (\tau_i, \xi_i)_{i=1}^{\infty} \in \mathbb{V}$, we get $\lambda \le J(x, V)$. Since we consider the minimisation problem, we can restrict our attention to the strategies for which

$$\mathbb{E}_{(x,V)}\left[e^{\sum_{i=1}^{\infty} 1_{\{\tau_i \le T\}} c(X^i_{\tau_i}, \xi_i)}\right] < \infty, \quad T \ge 0. \tag{4.3.15}$$

Using Proposition 4.3.2 and Doob's optional stopping theorem, we get that the process $e^{\int_0^{\tau_1\wedge T}(f(X^1_s)-\lambda)ds + u(X^1_{\tau_1\wedge T})}$, $T \ge 0$, is a $\mathbb{P}_{(x,V)}$-submartingale. Thus, for any $T \ge 0$, we get

$$e^{u(x)} \le \mathbb{E}_{(x,V)}\left[e^{\int_0^{\tau_1\wedge T}(f(Y_s)-\lambda)ds + u(X^1_{\tau_1\wedge T})}\right].$$

Using the fact that $u(X^n_{\tau_n}) \le Mu(X^n_{\tau_n}) \le c(X^n_{\tau_n}, \xi_n) + u(\xi_n)$ on $\{\tau_n < \infty\}$, as in (4.3.13), we get

$$\begin{aligned}e^{u(x)} \le \mathbb{E}_{(x,V)}\Big[&e^{\int_0^{\tau_n\wedge T}(f(Y_s)-\lambda)ds + \sum_{i=1}^{n} 1_{\{\tau_i \le T\}} c(X^i_{\tau_i}, X^{i+1}_{\tau_i})} \times\\ &\times e^{\sum_{i=0}^{n-1} 1_{\{\tau_i \le T < \tau_{i+1}\}} u(X^{i+1}_T) + 1_{\{\tau_n \le T\}} u(X^n_{\tau_n})}\Big].\end{aligned}$$

Recalling (4.3.15) and letting $n \to \infty$, we get

$$e^{u(x)} \le \mathbb{E}_{(x,V)}\left[e^{\int_0^T (f(Y_s)-\lambda)ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \le T\}} c(X^i_{\tau_i}, X^{i+1}_{\tau_i}) + \|u\|}\right].$$

As in Step 1, we get $\lambda \le J(x, V)$, $x \in E$, which concludes the proof. □

4.3.2 Existence of a solution to the Bellman equation

The solution to (4.3.3) is constructed using a dyadic approximation argument. Recall the time-step $\delta_m = \frac{1}{2^m}$, $m \in \mathbb{N}$, and the corresponding family of dyadic stopping times $\mathcal{T}^m_{x,b}$, $m \in \mathbb{N}$, $x \in E$, introduced in Section 2.1. For any $m \in \mathbb{N}$, the dyadic version of the Bellman equation (4.3.3) is given by

$$u_m(x) = \inf_{\tau \in \mathcal{T}^m_{x,b}} \ln \mathbb{E}_x \left[\exp \left(\int_0^\tau (f(X_s) - \lambda_m) ds + M u_m(x_\tau) \right) \right], \quad x \in E, \tag{4.3.16}$$

where M is defined in (4.3.4), and we want to find $u_m \in \mathcal{C}_b(E)$ and $\lambda_m \in \mathbb{R}$ satisfying (4.3.16). In fact, due to the dyadic nature of the problem, one could consider the associated one-step equation given by

$$e^{u_m(x)} = \min \left(e^{M u_m(x)}, \mathbb{E}_x \left[e^{\int_0^{\delta_m} (f(X_s) - \lambda_m) ds + u_m(X_{\delta_m})} \right] \right), \quad x \in E; \tag{4.3.17}$$

see Proposition 4.3.6 for details. In the following theorem we show that a solution to (4.3.17) exists and solves a dyadic version of the impulse control problem given by

$$\inf_{V \in \mathbb{V}^m} J(x, V), \quad x \in E. \tag{4.3.18}$$

Theorem 4.3.5. *For any $m \in \mathbb{N}$, there exists a function $u_m \in \mathcal{C}_b(E)$ and a constant $\lambda_m \in \mathbb{R}$ satisfying (4.3.17). Moreover, we get*

$$\lambda_m = \inf_{V \in \mathbb{V}^m} J(x, V), \quad x \in E.$$

Proof. Using Proposition A.3.3, we get that there exists a function $u_m \in \mathcal{C}_b(E)$ and a constant $\lambda_m \in \mathbb{R}$ satisfying

$$\begin{aligned} e^{u_m(x)} = \min \Bigg(&\inf_{\xi \in U} e^{c(x,\xi)} \, \mathbb{E}_\xi \left[e^{\int_0^{\delta_m} (f(X_s) - \lambda_m) ds + u_m(X_{\delta_m})} \right], \\ &\mathbb{E}_x \left[e^{\int_0^{\delta_m} (f(X_s) - \lambda_m) ds + u_m(X_{\delta_m})} \right] \Bigg). \end{aligned} \tag{4.3.19}$$

Let us show that

$$\inf_{\xi \in U} e^{c(x,\xi)} \, \mathbb{E}_\xi \left[e^{\int_0^{\delta_m} (f(X_s) - \lambda_m) ds + u_m(X_{\delta_m})} \right] = e^{M u_m(x)}, \tag{4.3.20}$$

which directly implies that the pair (u_m, λ_m) satisfy (4.3.17). For brevity, we define

$$F(x) := \mathbb{E}_x \left[e^{\int_0^{\delta_m} (f(X_s) - \lambda_m) ds + u_m(X_{\delta_m})} \right], \quad x \in E.$$

From (4.3.19), we get $F(x) \geq e^{u_m(x)}$, $x \in E$. Thus, for any $x \in E$, we get

$$\inf_{\xi \in U} e^{c(x,\xi)} F(\xi) \geq \inf_{\xi \in U} e^{c(x,\xi)} e^{u_m(\xi)} = e^{M u_m(x)}. \tag{4.3.21}$$

On the other hand, using (4.1.2) and (4.3.19), for any $x \in E$, we get

$$\begin{aligned}\inf_{\xi\in U} e^{c(x,\xi)}F(\xi) &= \inf_{\xi\in U} e^{c(x,\xi)}F(\xi) \wedge \inf_{\zeta\in U} e^{c(x,\zeta)}F(\zeta)\\ &\leq \inf_{\xi\in U}\left(e^{c(x,\xi)}F(\xi) \wedge \inf_{\zeta\in U} e^{c(x,\xi)}e^{c(\xi,\zeta)}F(\zeta)\right)\\ &= \inf_{\xi\in U} e^{c(x,\xi)}\left(F(\xi)\wedge \inf_{\zeta\in U} e^{c(\xi,\zeta)}F(\zeta)\right)\\ &= \inf_{\xi\in U} e^{c(x,\xi)}e^{u_m(\xi)} = e^{Mu_m(x)},\end{aligned}$$

which proves (4.3.20). Combining (4.3.19) and (4.3.20), we get that the pair (u_m, λ_m) is a solution to (4.3.17). Finally, using Proposition A.3.3 again, for any $x \in E$, we get $\lambda_m = \inf_{V\in\mathbb{V}^m} J(x, V)$, which concludes the proof. □

Now, we link a solution to (4.3.17) with (4.3.16) and (4.3.18). The link is based on the type of the semigroup $r(f)$ given by (4.1.8).

Proposition 4.3.6. *For any $m \in \mathbb{N}$, let $u_m \in \mathcal{C}_b(E)$ and $\lambda_m \in \mathbb{R}$ be a solution to* (4.3.17)*. Then, for any $m \in \mathbb{N}$, we get $\lambda_m \leq r(f)$. Moreover:*

(1) *If $\lambda_m = r(f)$, then the no impulse strategy is optimal for* (4.3.18)*.*

(2) *If $\lambda_m < r(f)$, then the pair (u_m, λ_m) satisfies* (4.3.16)*.*

Proof. Let us fix $m \in \mathbb{N}$. First, we show that we get $\lambda_m \leq r(f)$. Recalling $\mathcal{P}_t^f$ from (2.1.5) and using (A.4.2), for any $x \in E$ and the *no impulse* strategy $V_0 \in \mathbb{V}$, we get

$$J(x, V_0) = \limsup_{T\to\infty} \frac{1}{T} \ln \mathcal{P}_t^f \mathbb{1}(x) \leq \limsup_{T\to\infty} \frac{1}{T} \sup_{y\in E} \ln \mathcal{P}_t^f \mathbb{1}(y) = r(f).$$

Also, using Theorem 4.3.5, for λ_m we get

$$\lambda_m = \inf_{V\in\mathbb{V}_m} J(x, V) \leq J(x, V_0) \leq r(f), \tag{4.3.22}$$

which concludes the proof of $\lambda_m \leq r(f)$.

Now, suppose that $\lambda_m = r(f)$. From (4.3.22), we get $r(f) = J(x, V_0)$. This implies the optimality of the *no impulse* strategy.

Next, suppose that $\lambda_m < r(f)$. To show that (u_m, λ_m) satisfies (4.3.16), we use the change of measure technique based on the Multiplicative Poisson Equation. In this way, as in Proposition 4.3.2, we can replace the term $(f(\cdot) - \lambda_m)$ in (4.3.17) by some positive constant and use the results from

Chapter 3. Recalling v from ($\mathcal{B}$5b), u_m from (4.3.17), and ($\mathbb{Q}_x$) from (A.4.7), for any $m \in \mathbb{N}$ and $x \in E$, we set

$$\begin{aligned} w_m(x) &:= u_m(x) - v(x) + \|u_m\| + \|v\| + \|Mu_m\|, \\ G_m(x) &:= Mu_m(x) - v(x) + \|u_m\| + \|v\| + \|Mu_m\|, \\ S^m h(x) &:= \min\left(e^{G_m(x)}, \mathbb{E}_x^{\mathbb{Q}}\left[e^{(r(f)-\lambda_m)\delta_m} h(X_{\delta_m})\right]\right), \quad h \in \mathcal{C}_b^+(E). \end{aligned}$$

Using (A.4.8) and (4.3.17), we get

$$e^{w_m(x)} = S^m e^{w_m}(x), \quad x \in E. \tag{4.3.23}$$

Also, noting that $G_m \in \mathcal{C}_b^+(E)$ and $r(f) - \lambda_m > 0$, and using Theorem 3.2.15, we get

$$e^{w_m(x)} = \inf_{\tau \in \mathcal{T}_{x,b}^m} \mathbb{E}_x^{\mathbb{Q}}\left[e^{(r(f)-\lambda_m)\tau + G_m(X_\tau)}\right];$$

note that here we used (4.3.8) and the fact that, for any $x \in E$, the families of $\mathbb{P}_x$ and $\mathbb{Q}_x$ a.s. bounded stopping times coincide. Recalling (A.4.8) and the definitions of w_m and G_m, we conclude that u_m satisfies (4.3.16). □

Remark 4.3.7. In Proposition 4.3.6, to link a solution to the one-step equation (4.3.17) with the dyadic Bellman equation (4.3.16), we used the condition $\lambda_m < r(f)$. This, combined with the change of measure transformation, facilitates the use of Theorem 3.2.15. Using the same argument for $\lambda = r(f)$, we get

$$e^{w_m(x)} = \min\left(e^{G_m(x)}, \mathbb{E}_x^{\mathbb{Q}}\left[e^{w_m(X_{\delta_m})}\right]\right), \quad x \in E, \tag{4.3.24}$$

see the argument leading to (4.3.23) for details. In particular, there is no running cost function and the results from Section 3.2 are not applicable. However, (4.3.24) could be embedded in the framework of classic optimal stopping problems considered in Shiryaev (1978). In particular, using Theorem 15 from Chapter I of Shiryaev (1978), we get that w_m satisfying (4.3.24) could be represented as

$$e^{w_m(x)} = \inf_{\tau \in \mathcal{T}_x^m} \mathbb{E}_x^{\mathbb{Q}}\left[e^{G_m(X_\tau)}\right], \quad x \in E,$$

if and only if $\liminf_{n\to\infty} w_m(X_{n\delta_m}) = \liminf_{n\to\infty} G_m(X_{n\delta_m})$ $\mathbb{Q}_x$ a.s. Unfortunately, our setting does not allow for a direct verification of this condition. Instead, for the case $\lambda_m = r(f)$, we use the results from the theory of finite time horizon impulse control problems; see Theorem 4.3.9 for details. ◊

Let (λ_m), $m \in \mathbb{N}$, be a sequence of constants that corresponds to the solutions (u_m, λ_m) to the dyadic Bellman equations (4.3.16). Using Theorem 4.3.5, we know that this sequence exists and, in fact, it is unique as the values of the dyadic impulse control problems. Also, we get that (λ_m) is decreasing and $\lambda_m \geq -\|f\|$. Thus, we may define the finite limit

$$\lambda := \lim_{m\to\infty} \lambda_m. \tag{4.3.25}$$

From Proposition 4.3.6 we know that $\lambda \leq r(f)$. In Theorem 4.3.8 we show that if $\lambda < r(f)$, then there exists a solution to (4.3.3).

Theorem 4.3.8. *Let λ be given by (4.3.25). If $\lambda < r(f)$, then there exists a map $u \in \mathcal{C}_b(E)$, such that the pair (u, λ) is a solution to (4.3.3).*

Proof. For transparency, we split the proof into two steps: (1) proof of the fact that $(Mu_{m_n}) \to \phi$ uniformly to some $\phi \in \mathcal{C}_b(E)$, along a subsequence (m_n); (2) proof of the identity $\phi = Mu$ for a suitable u. In the end, we comment how to combine these steps to get (4.3.3) and conclude the proof.

Step 1. We show that, using the Arzelà–Ascoli theorem, one can choose a uniformly convergent subsequence of (Mu_m). First, we show that (Mu_m) is uniformly bounded on E. Note that we may choose a sequence (u_m) in such a way that, for any $m \in \mathbb{N}$ and $\xi \in U$, we get $u_m(\xi) \geq 0$. Indeed, for a generic $\widetilde{u}_m \in \mathcal{C}_b(E)$, $m \in \mathbb{N}$, which is a solution to the Bellman equation (4.3.17), we may set

$$u_m(x) := \widetilde{u}_m(x) - \inf_{\xi\in U} \widetilde{u}_m(\xi), \quad m \in \mathbb{N},\, x \in E; \tag{4.3.26}$$

note that $(\widetilde{u}_m)$ exists due to Theorem 4.3.5 and (u_m) also satisfies (4.3.17). Consequently, from (4.3.26) and the non-negativity of c, for any $m \in \mathbb{N}$ and $x \in E$, we get $Mu_m(x) \geq 0$, i.e. we have found the uniform lower bound for (Mu_m). Also, recalling (4.3.26), for any $m \in \mathbb{N}$ and $x \in E$, we get

$$Mu_m(x) = M\widetilde{u}_m(x) - \inf_{\xi\in U} \widetilde{u}_m(\xi) \leq \left(\|c\| + \inf_{\xi\in U} \widetilde{u}_m(\xi)\right) - \inf_{\xi\in U} \widetilde{u}_m(\xi) = \|c\|,$$

and consequently we get $0 \leq Mu_m(x) \leq \|c\|$, $m \in \mathbb{N}$, $x \in E$.

Second, we show that the family (Mu_m) is equicontinuous. In fact, this follows directly from the inequality

$$|Mu_m(x) - Mu_m(y)| \leq \sup_{\xi\in U} |c(x,\xi) - c(y,\xi)|, \quad m \in \mathbb{N},\, x, y \in E \tag{4.3.27}$$

and the continuity of the shift-cost function c. Thus, using the Arzelà–Ascoli theorem, for any $N \in \mathbb{N}$ and the compact set $B(N) := \{x \in E \colon \|x\| \leq N\}$, we can find a subsequence of $(Mu_m)_{m\in\mathbb{N}}$, say $(Mu_{m_n^N})_{n\in\mathbb{N}}$, and $\phi_N \in \mathcal{C}_b(E)$ such that

$$Mu_{m_n^N}(x) \to \phi_N(x), \quad n \to \infty, \tag{4.3.28}$$

uniformly in $x \in B(N)$.

Third, using a diagonal argument, we show that the limit may be chosen independently of N. Indeed, using recursive procedure and taking consecutive subsequences $\{m_n^N : n \in \mathbb{N}\} \subseteq \{m_n^{N-1} : n \in \mathbb{N}\}$, we can find a sequence of functions $(\phi_N)_{N\in\mathbb{N}}$, such that $\phi_N(x) = \phi_{N-1}(x)$ for $x \in B(N-1)$. Next, for any $x \in E$, we set $N_x := \inf\{N \in \mathbb{N} : x \in B(N)\}$ and define $\phi \in \mathcal{C}_b(E)$ by $\phi(x) := \phi_{N_x}(x)$. Also, for the diagonal sequence (m_n) given by $m_n := m_n^n$, $n \in \mathbb{N}$, we get

$$Mu_{m_n}(x) \to \phi(x), \quad n \to \infty, \tag{4.3.29}$$

uniformly in $x \in B(N)$, for any $N \in \mathbb{N}$. Also, using (4.3.27), we get that ϕ satisfies

$$|\phi(x) - \phi(y)| \le \sup_{\xi\in U} |c(x,\xi) - c(y,\xi)|, \quad x, y \in E. \tag{4.3.30}$$

Finally, we show that the convergence $Mu_{m_n} \to \phi$ is (globally) uniform. Let $\varepsilon > 0$. From (4.1.3), we get that there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sup_{\xi\in U} |c(x,\xi) - c(y,\xi)| \le \frac{\varepsilon}{3}, \quad x, y \notin B(N_\varepsilon). \tag{4.3.31}$$

Since $Mw_{m_n}(x) \to \phi(x)$ uniformly in $x \in B(N_\varepsilon + 1)$, it is sufficient to show that

$$\sup_{x\notin B(N_\varepsilon)} |Mu_{m_n}(x) - \phi(x)| \to 0, \quad n \to \infty. \tag{4.3.32}$$

Let $x \notin B(N_\varepsilon)$ and $y \in B(N_\varepsilon + 1) \setminus B(N_\varepsilon)$. Recalling (4.3.27) and (4.3.30), we get

$$\begin{aligned} |Mu_{m_n}(x) - \phi(x)| &\le |Mu_{m_n}(x) - Mu_{m_n}(y)| + |Mu_{m_n}(y) - \phi(y)| \\ &\quad + |\phi(y) - \phi(x)| \\ &\le |Mu_{m_n}(y) - \phi(y)| + 2\sup_{\xi\in U} |c(x,\xi) - c(y,\xi)|. \end{aligned}$$

Since $y \in B(N_\varepsilon+1)$, starting from some $n_0 \in \mathbb{N}$, we get $|Mu_{m_n}(y) - \phi(y)| \le \frac{\varepsilon}{3}$ for $n \ge n_0$. This, combined with (4.3.31), shows that for $n \ge n_0$, we get

$$\sup_{x\notin B(N_\varepsilon)} |Mu_{m_n}(x) - \phi(x)| \le \varepsilon.$$

Thus, we get (4.3.32), which concludes Step 1 of the proof.

Step 2. For brevity, we suppress the subscript n from the diagonal sequence (m_n) given in Step 1, i.e. we assume that $Mu_m(x) \to \phi(x)$ as $m \to \infty$ uniformly in $x \in E$. We show that

$$\phi(x) = Mu(x), \quad x \in E, \tag{4.3.33}$$

where $u\colon E \to \mathbb{R}$ is given by

$$e^{u(x)} := \inf_{\tau \in \mathcal{T}_{x,b}} \mathbb{E}_x \left[e^{\int_0^\tau (f(X_s)-\lambda)ds + \phi(X_\tau)} \right]; \tag{4.3.34}$$

note that this shows the existence of a solution to (4.3.3).

As in the proof of Proposition 4.3.6, we use the change of measure technique to transform (4.3.34) into the setting where the assumptions of Theorem 3.5.3 are satisfied. For any $m \in \mathbb{N}$ and $x \in E$, let us define

$$\begin{aligned}
G_m(x) &:= Mu_m(x) - v(x) + \|Mu_m\| + \|v\|,\\
G(x) &:= \phi(x) - v(x) + \|\phi\| + \|v\|,\\
d_m &:= r(f) - \lambda_m,\\
d &:= r(f) - \lambda,\\
w^m(x) &:= u_m(x) - v(x) + \|Mu_m\| + \|v\|,\\
w(x) &:= u(x) - v(x) + \|\phi\| + \|v\|.
\end{aligned}$$

where v is a solution to (4.1.7). Note that $G_m \to G$ uniformly and $d_m \uparrow d$. Moreover, recalling that we assumed $\lambda < r(f)$, we get $d > 0$. Next, using Proposition 4.3.6, we get that (u_m, λ_m) is a solution to (4.3.16). Thus, using (A.4.8), we get

$$w^m(x) = \inf_{\tau \in \mathcal{T}^m_{x,b}} \ln \mathbb{E}^{\mathbb{Q}}_x \left[e^{d_m \tau + G_m(X_\tau)} \right], \quad m \in \mathbb{N},\, x \in E.$$

Similarly, using (4.3.34) and (A.4.8), we get

$$w(x) = \inf_{\tau \in \mathcal{T}_{x,b}} \ln \mathbb{E}^{\mathbb{Q}}_x \left[e^{d\tau + G(X_\tau)} \right], \quad x \in E.$$

Hence, using Theorem 3.5.3, we get $w^m(x) \to w(x)$ as $m \to \infty$ uniformly in x from compact sets. Thus, recalling uniform convergence of Mu_m to ϕ, we get $u_m(x) \to u(x)$ as $m \to \infty$ uniformly in x from compact sets. Moreover, using Theorem 3.4.11, we get $w \in \mathcal{C}_b(E)$ and consequently $u \in \mathcal{C}_b(E)$. Finally, recalling (4.3.4), we get

$$\sup_{x\in E} |Mu_m(x) - Mu(x)| \leq \sup_{\xi\in U} |u_m(\xi) - u(\xi)| \to 0, \quad m \to \infty.$$

This implies uniform convergence of Mu_m to Mu. Recalling that, from Step 1, we know that $Mu_m \to \phi$, we get $\phi(x) = Mu(x)$, $x \in E$. Recalling (4.3.34), we conclude the proof. $\square$

Finally, we are ready to link the constant λ given by (4.3.25) to an optimal strategy and the optimal value of (4.3.1); see Theorem 4.3.9. In the case $\lambda < r(f)$, Theorem 4.3.8 guarantees the existence of a solution to the Bellman equation (4.3.3), which, combined with Theorem 4.3.4, gives the optimal value and an optimal impulse control strategy for (4.3.1). In the degenerate case $\lambda = r(f)$, due to the monotonicity of (λ_m), we know that $\lambda_m = r(f)$ for any $m \in \mathbb{N}$. Thus, from Proposition 4.3.6, we get the *no impulse* strategy is optimal for any dyadic time-grid. As expected, we show that this implies the optimality of the *no impulse* strategy also in the continuous time setting. Here we make an additional assumption $E = U$, which allows us to use finite time horizon results from Proposition 4.2.9.

Theorem 4.3.9. *Let λ be given by (4.3.25). Then:*

(1) *If $\lambda < r(f)$, then $\lambda = \inf_{V\in\mathbb{V}} J(x, V)$ for any $x \in E$, and the strategy defined in (4.3.10) via the Bellman equation (4.3.3) is optimal.*

(2) *If $\lambda = r(f)$ and $E = U$, then $\lambda = \inf_{V\in\mathbb{V}} J(x, V)$ for any $x \in E$, and the no impulse strategy is optimal.*

Proof. Suppose that $\lambda < r(f)$. Then, from Theorem 4.3.8, we get that there exists a solution (u, λ) to (4.3.3). Thus, using Theorem 4.3.4, we get the optimal value and an optimal strategy for (4.3.1).

Now, suppose that $\lambda = r(f)$. Using Theorem 4.3.5 and Proposition 4.3.6, we get that the cost of the *no impulse* strategy equals $r(f)$. Thus, it is sufficient to show that, in this case, for any $x \in E$, we get $\inf_{V\in\mathbb{V}} J(x, V) \geq r(f)$. For the contradiction, suppose that $\inf_{V\in\mathbb{V}} J(x_0, V) < r(f)$ for some $x_0 \in E$. Then, for some $\varepsilon > 0$, we get

$$\limsup_{T\to\infty} \inf_{V\in\mathbb{V}} \frac{1}{T} J_T(x_0, V) \leq \inf_{V\in\mathbb{V}} J(x_0, V) < r(f) - \varepsilon, \tag{4.3.35}$$

where J_T is given by (4.2.1). Next, we can find $T_0 \in \mathbb{N}$ big enough to get

$$\inf_{V\in\mathbb{V}} \frac{1}{T_0} J_{T_0}(x_0, V) \leq \limsup_{T\to\infty} \inf_{V\in\mathbb{V}} \frac{1}{T} J_T(x_0, V) + \frac{\varepsilon}{4} \quad \text{and} \quad \frac{\|c\|}{T_0} \leq \frac{\varepsilon}{4}. \tag{4.3.36}$$

Using Propostion 4.2.9, we can find $n \in \mathbb{N}$, $m \in \mathbb{N}$, and a strategy $\bar{V} \in \mathbb{V}_n^m$, such that

$$\frac{1}{T_0} J_{T_0}(x_0, \bar{V}) \leq \inf_{V \in \mathbb{V}} \frac{1}{T_0} J_{T_0}(x_0, V) + \frac{\varepsilon}{4}. \tag{4.3.37}$$

We define the strategy $\widetilde{V}$ in the following way: for any period $[kT_0, (k+1)T_0]$, $k \in \mathbb{N}$, we follow the strategy $\bar{V}$ and at $(k+1)T_0$ we shift the process to x_0. Since $E = U$ and $T_0 \in \mathbb{N}$, for any $m \in \mathbb{N}$, we get $\widetilde{V} \in \mathbb{V}^m$. Then, we get

$$\begin{aligned} &J(x_0, \widetilde{V}) \\ &\quad = \limsup_{k \to \infty} \frac{1}{kT_0} \ln \left(\mathbb{E}_{(x_0, \widetilde{V})} \left[e^{\int_0^{T_0} f(Y_s) ds + \sum_{i=1}^{n} 1_{\{\tau_i \leq T_0\}} c(Y_{\tau_i}^-, \xi_i) + c(Y_{T_0}, x_0)} \right] \right)^k \\ &\qquad\qquad\qquad\qquad \leq \frac{1}{T_0} J_{T_0}(x_0, \bar{V}) + \frac{\|c\|}{T_0}. \end{aligned} \tag{4.3.38}$$

Combining (4.3.35)–(4.3.37) with (4.3.38), we get

$$J(x_0, \widetilde{V}) < r(f) - \frac{\varepsilon}{4}. \tag{4.3.39}$$

Recalling that, by Proposition 4.3.6, we get $\inf_{V \in \mathbb{V}^m} J(x_0, V) = r(f)$, $m \in \mathbb{N}$, we conclude that (4.3.39) leads to the contradiction. $\square$

Chapter 5

Reference examples and further remarks

In this chapter, we provide examples and further remarks related to the conditions and results considered in this thesis. In particular, we comment on the sets of assumptions from Chapter 3 and Chapter 4. We verify them for several classes of Markov processes and show how to effectively embed some practical dynamics into the framework considered in this thesis. Also, in this chapter we provide specific computable examples related to the optimal stopping Bellman equation, both in the discrete and continuous time settings. In particular, using the results from Chapter 3, we provide explicit formulae for the value functions of the suitable stopping problems and show that the optimal stopping Bellman equation may admit multiple solutions.

The structure of this chapter is as follows. In Section 5.1, we comment on the optimal stopping framework. Next, in Section 5.2, we discuss several examples that could be embedded in the impulse control setting. Finally, in Section 5.3, we discuss computable toy examples related to the optimal stopping Bellman equation.

The results presented in this chapter are based on Section 5 of Jelito et al. (2020) and Section 6 of Jelito and Stettner (2022). However, it should be noted that we substantially expanded the analysis of some examples. In particular, in Example 5.2.1, we provide more detailed comments on the process construction and the fact that the resulting dynamics satisfies the necessary conditions. Also, in Example 5.3.5, we provide a detailed discussion on the verification of the model assumptions and the construction of multiple solutions to the Bellman equation.

5.1 Discussion on the optimal stopping framework

In this section, we discuss in more detail selected assumptions from Section 3.1 and we provide exemplary dynamics that could be embedded in that framework. More specifically, we focus on ($\mathcal{A}2$)–($\mathcal{A}3$) and give a generic integrability condition that facilitates the verification of these assumptions. Note that the remaining conditions from Section 3.1 are relatively easy to verify. In particular, Assumption ($\mathcal{A}1$) only requires some regularity properties of the cost functions while Assumption ($\mathcal{A}4$) could be derived e.g. from the $\mathcal{C}_0$-Feller property; see Proposition 2.1.7 for details.

Throughout this section, by $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ we denote a continuous time standard $\mathcal{C}_b$-Feller-Markov process and we assume that the maps g and G satisfies ($\mathcal{A}1$). Also, for any $T\geq 0$, we define $\zeta_T := \sup_{t\in[0,T]} e^{G(X_t)}$. In Lemma 5.1.1 below we show that ($\mathcal{A}2$) and ($\mathcal{A}3$) could be derived from the following condition:

(UI) (Uniform integrability). For any $T\geq 0$ and a compact set $K\subseteq E$, we have

$$\lim_{m\to\infty}\sup_{x\in K}\mathbb{E}_x\left[\zeta_T 1_{\{\zeta_T\geq m\}}\right]=0.$$

This condition could be seen as a stronger form of the integrability of ζ_T, $T\geq 0$. Namely, it requires that the tail of ζ_T, $T\geq 0$, is $\mathbb{P}_x$-integrable uniformly in x from a compact set. Exemplary dynamics satisfying (UI) is shown in Example 5.1.2.

Lemma 5.1.1. *Suppose that (UI) is satisfied. Then, we get ($\mathcal{A}2$) and ($\mathcal{A}3$).*

Proof. For ($\mathcal{A}2$), let us fix some $T\geq 0$ and $x\in E$. Using (UI) with $K:=\{x\}$, we may find $m_0\in\mathbb{N}$ such that, for any $m\geq m_0$, we get $\mathbb{E}_x\left[\zeta_T 1_{\{\zeta_T\geq m\}}\right]\leq 1$. Thus, for $m\geq m_0$, we get

$$\mathbb{E}_x\left[\zeta_T\right]=\mathbb{E}_x\left[\zeta_T 1_{\{\zeta_T< m\}}\right]+\mathbb{E}_x\left[\zeta_T 1_{\{\zeta_T\geq m\}}\right]\leq m+1<\infty,$$

which shows ($\mathcal{A}2$).

For ($\mathcal{A}3$), let $T\geq 0$, $x\in E$, $(x_n)\to x\in E$, and $h\in\mathcal{C}^+(E)$ be such that $h(\cdot)\leq G(\cdot)$. Let $\Gamma\subseteq E$ be a compact set satisfying $x\in\Gamma$ and $(x_n)\subset\Gamma$. Then, we get

$$\begin{aligned}
&\left|\mathbb{E}_x\left[e^{\int_0^T g(X_s)ds+h(X_T)}\right]-\mathbb{E}_{x_n}\left[e^{\int_0^T g(X_s)ds+h(X_T)}\right]\right|\\
&\quad\leq\left|\mathbb{E}_x\left[e^{\int_0^T g(X_s)ds+h(X_T)\wedge m}\right]-\mathbb{E}_{x_n}\left[e^{\int_0^T g(X_s)ds+h(X_T)\wedge m}\right]\right|\\
&\qquad+2\sup_{y\in\Gamma}\left|\mathbb{E}_y\left[e^{\int_0^T g(X_s)ds+h(X_T)}\right]-\mathbb{E}_y\left[e^{\int_0^T g(X_s)ds+h(X_T)\wedge m}\right]\right|.
\end{aligned}$$

Also, using the $\mathcal{C}_b$-Feller property and Proposition 2.1.8, we get that the map

$$x \mapsto \mathbb{E}_x \left[e^{\int_0^T g(X_s)ds + h(X_T)\wedge m} \right]$$

is continuous for any $m \in \mathbb{N}$. Thus, to conclude the proof, it is enough to show that

$$\sup_{y\in\Gamma} \left| \mathbb{E}_y \left[e^{\int_0^T g(X_s)ds + h(X_T)} \right] - \mathbb{E}_y \left[e^{\int_0^T g(X_s)ds + h(X_T)\wedge m} \right] \right| \to 0, \quad m \to \infty. \tag{5.1.1}$$

Recalling that $h(\cdot) \leq G(\cdot)$ and using (UI), we get

$$\begin{aligned} \sup_{y\in\Gamma} \mathbb{E}_y & \left[e^{\int_0^T g(X_s)ds} \left| e^{h(X_T)} - e^{h(X_T)\wedge m} \right| \right] \\ & \leq 2 \sup_{y\in\Gamma} \mathbb{E}_y \left[e^{\int_0^T g(X_s)ds} e^{h(X_T)} 1_{\{h(X_T)\geq m\}} \right] \\ & \leq 2e^{T\|g\|} \sup_{y\in\Gamma} \mathbb{E}_y \left[\zeta_T 1_{\{\zeta_T \geq e^m\}} \right] \to 0, \quad m \to \infty. \end{aligned}$$

Thus, we get (5.1.1), which concludes the proof. □

Now, we show how to verify Condition (UI) in a simple Brownian motion model; see Example 5.1.2. Note that another application of (UI) can be found in Example 5.3.5.

Example 5.1.2. Let $E := \mathbb{R}$, $G(x) := |x|$, and the process (X_t) be a Brownian motion. Note that under $\mathbb{P}_x$ we get $X_t = x + W_t$, where W is a standard Brownian motion (starting from 0). For simplicity, for any $T \geq 0$, we set $\zeta_T := \sup_{t\in[0,T]} e^{|X_t|}$ and $S_T := \sup_{t\in[0,T]} |W_t|$, $T \geq 0$. Let $K \subseteq E$ be a compact set and let $T \geq 0$. We show that we get

$$\lim_{m\to\infty} \sup_{x\in K} \mathbb{E}_x \left[\zeta_T 1_{\{\zeta_T \geq e^m\}} \right] = 0, \tag{5.1.2}$$

which implies (UI).

Note that, setting $L_K := \sup_{x\in K} |x|$, for any $m \in \mathbb{N}$, we get

$$\begin{aligned} \sup_{x\in K} \mathbb{E}_x \left[\zeta_T 1_{\{\zeta_T \geq e^m\}} \right] &= \sup_{x\in K} \mathbb{E}_x \left[\sup_{t\in[0,T]} e^{|x+W_t|} 1_{\{\sup_{t\in[0,T]} |x+W_t| \geq m\}} \right] \\ &\leq \sup_{x\in K} e^{|x|} \mathbb{E}_x \left[e^{S_T} 1_{\{S_T \geq m - L_K\}} \right]. \end{aligned} \tag{5.1.3}$$

Moreover, we get that $\mathbb{E}_x\left[e^{S_T}1_{\{S_T\geq m-L_K\}}\right]$ is independent of $x\in E$. Thus, noting that $\sup_{x\in K} e^{|x|}<\infty$, to conclude the proof of (5.1.2), it is enough to show

$$\mathbb{E}_0\left[e^{S_T}\right]<\infty. \tag{5.1.4}$$

Indeed, noting that $\mathbb{E}_0\left[e^{S_T}1_{\{S_T<m-L_K\}}\right]$ converges increasingly to $\mathbb{E}_0\left[e^{S_T}\right]$ as $m\to\infty$, using the identity

$$\mathbb{E}_0\left[e^{S_T}\right]=\mathbb{E}_0\left[e^{S_T}1_{\{S_T<m-L_K\}}\right]+\mathbb{E}_0\left[e^{S_T}1_{\{S_T\geq m-L_K\}}\right],\quad m\in\mathbb{N},$$

and recalling (5.1.4), we get $\lim_{m\to\infty}\mathbb{E}_0\left[e^{S_T}1_{\{S_T\geq m-L_K\}}\right]=0$, This, together with (5.1.3), implies (5.1.2).

Let us now show (5.1.4). Noting that $(-W)$ is also a standard Brownian motion, we get

$$\mathbb{E}_0\left[e^{S_T}\right]\leq\mathbb{E}_0\left[e^{\max\left(\sup_{t\in[0,T]}W_t,\sup_{t\in[0,T]}(-W_t)\right)}\right]\leq 2\mathbb{E}_0\left[e^{\sup_{t\in[0,T]}W_t}\right].$$

Recall that, by the reflection principle, the distribution of $\sup_{t\in[0,T]}W_t$ is equal to the distribution of $|W_T|$; see e.g. Proposition 3.7 in Chapter III of Revuz and Yor (1999) for details. Thus, we get $\mathbb{E}_0\left[e^{S_T}\right]\leq 2\mathbb{E}_0\left[e^{|W_T|}\right]<\infty$, which concludes the proof. ♦

5.2 Discussion on the impulse control framework

In this section, we discuss in more detail selected assumptions from Section 4.1. More specifically, we focus on Assumptions ($\mathcal{B}4$)–($\mathcal{B}5$) and directly verify them for two classes of continuous time Markov processes, i.e. piecewise deterministic processes and reflected diffusions (with possible jumps). The remaining conditions for the impulse control framework, i.e. Assumption ($\mathcal{B}1$)–($\mathcal{B}3$), are relatively easy to verify as they require the compactness of a suitable subspace of the state space or are related to the cost functions specification.

5.2.1 Piecewise deterministic Markov processes

The first class of dynamics satisfying ($\mathcal{B}4$)–($\mathcal{B}5$) takes the form of piecewise deterministic processes. In a nutshell, the process exhibits deterministic behaviour on the random time intervals and is subject to jumps at the exponential jump times. This class of processes was introduced in Davis (1984) and discussed in detail in Davis (1993); see also Bäuerle and Rieder (2011) for a more recent use in the optimisation context. Example 5.2.1 is partially based on Example 5.2 from Pitera and Stettner (2021).

Example 5.2.1. In this example $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$, is a piecewise deterministic Markov process with values in the state space $E := \mathbb{R}$ with the Borel σ-field $\mathcal{E}$. In a nutshell, under the measure $\mathbb{P}_x$, the process X starts at $x \in E$ and up to the exponentially distributed time T_1 follows some deterministic dynamics. At T_1, the process is subject to the immediate jump with a suitable jump measure. Then, the process starts its evolution at the after-jump state and the procedure repeats.

Let us now provide a more extensive comment on the process construction; details can be found in Davis (1993). First, let (T_n), $n \in \mathbb{N}$, be a sequence of random variables such that $T_0 \equiv 0$ and the increments $(T_{n+1} - T_n)$, $n \in \mathbb{N}$, are independent and identically distributed random variables following the exponential distribution with the rate parameter $\beta > 0$. The sequence (T_n), $n \in \mathbb{N}_*$, indicates the jump times of the process. Second, let $\mathbb{R}_+ \ni t \mapsto x_t^x \in E$, $x \in E$, be a solution to a (deterministic) stable differential equation starting in $x \in E$, i.e. we have

$$dx_t^x = F(x_t^x)dt, \quad x \in E, \tag{5.2.1}$$

with $x_0^x = x$ and a suitable map F. Assuming a sufficient regularity of F, we get $x_t^x = \phi(x, t)$, $x \in E$, $t \geq 0$, for some jointly continuous function ϕ. The maps (x_t^x) describe deterministic part of X. More specifically, we define the process (X_t) between the consecutive jumps as $X_t := \phi(X_{T_n}, t - T_n)$, $t \in (T_n, T_{n+1})$. At T_n, the process is subject to immediate jump. We assume that the jump measure for the process being at state $x \in E$ is the Gaussian measure given by

$$Q(x, A) := \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-B(x))^2}{2}} dy, \quad x \in E,\, A \in \mathcal{E},$$

where $B \in \mathcal{C}_b(E)$ is some fixed map describing the expected value of $Q(x, \cdot)$. In other words, we get

$$X_{T_n} = B(X_{T_n^-}) + \xi_n, \quad n \in \mathbb{N}_*,$$

where $X_{T_n^-}$ denotes the state of the process right-before the jump and (ξ_n), $n \in \mathbb{N}_*$, is a sequence of independent and identically distributed standard Gaussian random variables. Using Theorem 25.5 and Theorem 27.6 from Davis (1993), we find a family of measures $(\mathbb{P}_x)$, $x \in E$, such that $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a standard $\mathcal{C}_b$-Feller-Markov process with values in $(E, \mathcal{E})$. In particular, under this family of measures, for any $x \in E$, we get $\mathbb{P}_x[X_0 = x] = \mathbb{P}_x[x_0^x = x] = 1$.

Let us now show that the process X satisfies the assumptions from Section 4.1. For transparency, we split the argument into three steps: (1) proof that Assumption ($\mathcal{B}4$) is satisfied; (2) proof that Assumption ($\mathcal{B}5$a) is satisfied; (3) proof that Assumption ($\mathcal{B}5$b) is satisfied.

Step 1. We show that Assumption ($\mathcal{B}4$) is satisfied. For any $\Gamma \subset E$ and $t, r > 0$, let us define $M_\Gamma(t, r) := \sup_{x\in\Gamma} \mathbb{P}_x[\sup_{s\in[0,t]} |X_s - x| \geq r]$. First, we show that, for any compact set $\Gamma \subset E$ and $r_0 > 0$, we get

$$\lim_{t\to 0} M_\Gamma(t, r_0) = 0. \tag{5.2.2}$$

For the contradiction, suppose that there exist a compact set $\Gamma \subset E$, $r_0 > 0$, and $\varepsilon > 0$, such that, for any $n \in \mathbb{N}_*$, we may find $x_n \in \Gamma$ satisfying

$$\mathbb{P}_{x_n}\left[\sup_{s\in[0,\frac{1}{n}]} |X_s - x| \geq r_0\right] \geq \varepsilon > 0. \tag{5.2.3}$$

Next, recalling that T_1 is exponentially distributed (under any $\mathbb{P}_x$), we may find $n_0 \in \mathbb{N}_*$ such that, for any $n \geq n_0$, we get $\inf_{x\in\Gamma} \mathbb{P}_x\left[T_1 > \frac{1}{n}\right] \geq 1 - \frac{\varepsilon}{2}$. Combining this with (5.2.3), for any $n \geq n_0$, we get

$$\mathbb{P}_{x_n}\left[\sup_{s\in[0,\frac{1}{n}]} |X_s - x| \geq r_0, T_1 > \frac{1}{n}\right] \geq \frac{\varepsilon}{2}. \tag{5.2.4}$$

Next, note that, on the event $\{T_1 > \frac{1}{n}\}$, under the measure $\mathbb{P}_x$, we get $X_s = \phi(x, s)$, $s \in [0, \frac{1}{n}]$, $x \in E$. Also, using the joint continuity of ϕ and the compactness of Γ, we may find $n_1 \in \mathbb{N}_*$ such that, for any $n \geq n_1$, we get

$$\sup_{s\in[0,\frac{1}{n}]} \sup_{x\in\Gamma} |\phi(x, s) - x| < r_0.$$

Thus, for any $n \geq \max(n_0, n_1)$, we get

$$\begin{aligned}
&\mathbb{P}_x\left[\sup_{s\in[0,\frac{1}{n}]} |X_s - x| \geq r_0, T_1 > \frac{1}{n}\right] \\
&\qquad\qquad = \mathbb{P}_x\left[\sup_{s\in[0,\frac{1}{n}]} |\phi(x, s) - x| \geq r_0, T_1 > \frac{1}{n}\right] = 0, \quad x \in \Gamma,
\end{aligned}$$

which contradicts (5.2.4). Consequently, we get (5.2.2).

Second, we show that, for any compact set $\Gamma \subset E$ and $t_0 > 0$, we get

$$\lim_{r\to\infty} M_\Gamma(t_0, r) = 0. \tag{5.2.5}$$

For the contradiction, suppose that there exist a compact set $\Gamma \subset E$, $t_0 > 0$, and $\varepsilon > 0$, such that, for any $r > 0$, we get $M_\Gamma(t_0, r) \geq \varepsilon > 0$. Next, note that T_n, $n \in \mathbb{N}_*$, as the sum of the independent exponentially distributed random variables, follows the Erlang distribution with the shape parameter n and the rate parameter β. Hence, we we may find $n_0 \in \mathbb{N}_*$ such that, for any $x \in E$, we get $\mathbb{P}_x[T_{n_0} > t_0] \geq 1 - \frac{\varepsilon}{2}$. Thus, for any $r > 0$, we get

$$\sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t_0]} |X_s - x| \geq r, T_{n_0} > t_0 \right] \geq \frac{\varepsilon}{2}. \tag{5.2.6}$$

Noting that, for any $r > 0$, we get

$$\begin{aligned} &\sup_{x\in\Gamma} \mathbb{P}_x \left[\sup_{s\in[0,t_0]} |X_s - x| \geq r, T_{n_0} > t_0 \right] \\ &\qquad = \sup_{x\in\Gamma} \mathbb{P}_x \left[\bigcup_{k=0}^{n_0-1} \{t_0 \in [T_k, T_{k+1}), \sup_{s\in[0,t_0]} |X_s - x| \geq r, T_{n_0} > t_0\} \right] \\ &\qquad\qquad \leq \sum_{k=0}^{n_0-1} \sup_{x\in\Gamma} \mathbb{P}_x \left[t_0 \in [T_k, T_{k+1}), \sup_{s\in[0,t_0]} |X_s - x| \geq r \right], \end{aligned} \tag{5.2.7}$$

for the contradiction it is enough to show that we may find $R > 0$ such that, for any $k = 0, \ldots, n_0 - 1$, we get $P(k, R) \leq \frac{\varepsilon}{4n_0}$, where

$$P(k,r) := \sup_{x\in\Gamma} \mathbb{P}_x \left[t_0 \in [T_k, T_{k+1}), \sup_{s\in[0,t_0]} |X_s - x| \geq r \right], \quad r > 0.$$

Also, note that, for any $k = 0, \ldots, n_0 - 1$ and $r > 0$, we get

$$P(k,r) \leq \sum_{i=0}^{k} \sup_{x\in\Gamma} \mathbb{P}_x \left[t_0 \in [T_k, T_{k+1}), \sup_{s\in[T_i, T_{i+1}\wedge t_0)} |X_s - x| \geq r \right].$$

We show that, for any $k = 0, \ldots, n_0 - 1$ and $i = 0, \ldots, k$, we may find $R_k^i > 0$ such that $P_i(k, R_k^i) \leq \frac{\varepsilon}{4n_0^2}$, where

$$P_i(k,r) := \sup_{x\in\Gamma} \mathbb{P}_x \left[t_0 \in [T_k, T_{k+1}), \sup_{s\in[T_i, T_{i+1}\wedge t_0)} |X_s - x| \geq r \right], \quad r > 0.$$

Then, noting that, for any $k = 0, \ldots, n_0 - 1$, the map $r \mapsto P(k, r)$ is decreasing, and setting $R := \max\{R_k^i \colon k = 0, \ldots, n_0 - 1, i = 0, \ldots, k\}$, we get $P(k,R) \le \frac{\varepsilon}{4n_0}$, $k = 0,\dots,n_0-1$. This, combined with (5.2.7), shows $\sup_{x\in\Gamma}\mathbb{P}_x\left[\sup_{s\in[0,t_0]}|X_s - x| \ge r, T_{n_0} > t_0\right] \le \frac{\varepsilon}{4}$, which contradicts (5.2.6).

Now, let us fix $k = 0,\dots,n_0-1$ and $i = 0,\dots,k$, and show that, for some $R_k^i > 0$, we get

$$P_i(k, R_k^i) \le \frac{\varepsilon}{4n_0^2}. \tag{5.2.8}$$

For transparency, we use separate arguments for $i = 0$ and for $i > 0$. For $i = 0$, using the continuity of ϕ, we may find $R_k^0 > 0$ such that

$$\sup_{s\in[0,t_0]}\sup_{x\in\Gamma}|\phi(x,s) - x| < R_k^0.$$

Then, noting that

$$P_0(k, R_k^0) = \sup_{x\in\Gamma}\mathbb{P}_x\left[t_0 \in [T_k, T_{k+1}), \sup_{s\in[0,T_1\wedge t_0)}|\phi(x,s) - x| \ge R_k^0\right] = 0 \le \frac{\varepsilon}{4n_0^2},$$

we get (5.2.8) for $i = 0$. Next, we consider $i \in \{1,\dots,k\}$. Note that, using the Gaussianity of ξ_i, we may find a compact set $K_i \subset E$ such that, for any $x \in E$, we get

$$\mathbb{P}_x[\xi_i \in K_i] \ge 1 - \frac{\varepsilon}{4n_0^2}.$$

Also, using the boundedness of the map B, we may find a compact set $\Gamma_i \subset E$ such that

$$\{y_1 + y_2 \colon y_1 \in B(\mathbb{R}), y_2 \in K_i\} \subset \Gamma_i.$$

Next, using the continuity of ϕ, we may find $R_k^i > 0$ such that

$$\sup_{s\in[0,t_0]}\sup_{y\in\Gamma_i}|\phi(y,s)| + \sup_{z\in\Gamma}|z| < R_k^i.$$

In particular, on the event $\{t_0 \in [T_k, T_{k+1})\} \cap \{\xi_i \in K_i\}$, for any $x \in \Gamma$, we get

$$\begin{aligned}
\sup_{s\in[T_i,T_{i+1}\wedge t_0)}|X_s - x| &= \sup_{s\in[T_i,T_{i+1}\wedge t_0]}|\phi(X_{T_i}, s - T_i) - x| \\
&= \sup_{s\in[T_i,T_{i+1}\wedge t_0)}|\phi(B(X_{T_i^-}) + \xi_i, s - T_i) - x| \\
&= \sup_{s\in[0,t_0]}\sup_{y_1\in B(\mathbb{R})}\sup_{y_2\in K_i}|\phi(y_1 + y_2, s)| + \sup_{z\in\Gamma}|z| < R_k^i.
\end{aligned}$$

Thus, noting that

$$\begin{aligned}P_i(k,R_k^i) &\le \sup_{x\in\Gamma}\mathbb{P}_x\left[t_0\in[T_k,T_{k+1}),\sup_{s\in[T_i,T_{i+1}\wedge t_0)}|X_s-x|\ge R_k^i,\xi_i\notin K_i\right]\\&\quad+\sup_{x\in\Gamma}\mathbb{P}_x\left[t_0\in[T_k,T_{k+1}),\sup_{s\in[T_i,T_{i+1}\wedge t_0)}|X_s-x|\ge R_k^i,\xi_i\in K_i\right]\\&\le \sup_{x\in\Gamma}\mathbb{P}_x[\xi_i\notin K_i]+0\le\frac{\varepsilon}{4n_0^2},\end{aligned}$$

we get (5.2.8). Consequently, we get (5.2.5), which concludes the proof of this step.

Step 2. We show that ($\mathcal{B}$5a) is satisfied. Let us fix $x\in E$, $A\subset E$, and $t>0$. Also, to simplify the notation, we set $C^{-1}(s):=\{y\in E\colon \phi(y,s)\in C\}$, $C\in\mathcal{E}$, $s\ge 0$. Then, recalling that T_1 and T_2-T_1 are independent exponentially distributed random variables, we get

$$\begin{aligned}\mathbb{P}_x[X_t\in A]&\ge\mathbb{P}_x[t\in[T_1,T_2),X_t\in A]\\&=\mathbb{P}_x[t\in[T_1,T_2),\phi(X_{T_1},t-T_1)\in A]\\&=\mathbb{P}_x[t\in[T_1,T_2),\phi(B(x^x_{T_1})+\xi_1,t-T_1)\in A]\\&=\mathbb{P}_x[t\ge T_1,t-T_1<T_2-T_1,(B(x^x_{T_1})+\xi_1)\in A^{-1}(t-T_1)]\\&=\int_0^t\left(\int_{t-s_1}^\infty\beta e^{-\beta s_2}ds_2\right)Q(x^x_{s_1},A^{-1}(t-s_1))\beta e^{-\beta s_1}ds_1.\end{aligned}\tag{5.2.9}$$

Also, recalling that $Q(y,\cdot)$ is the Gaussian measure with mean $B(y)$ and using the fact that B is bounded, we may find a constant $a>0$ and a probability measure ν on $(E,\mathcal{E})$, such that $\nu(U)>0$ and $Q(y,C)\ge a\nu(C)$, $y\in E$, $C\in\mathcal{E}$. Thus, from (5.2.9), we get

$$\begin{aligned}\mathbb{P}_x[X_t\in A]&\ge a\int_0^t\left(\int_{t-s_1}^\infty\beta e^{-\beta s_2}ds_2\right)\mu(A^{-1}(t-s_1))\beta e^{-\beta s_1}ds_1\\&=a\int_0^t(1-e^{-\beta(t-s_1)})\nu(A^{-1}(t-s_1))\beta e^{-\beta s_1}ds_1.\end{aligned}$$

Thus, setting $\nu_t(C):=\frac{1}{a_t}\widetilde{\nu}_t(C)$, $C\in\mathcal{E}$, where

$$\widetilde{\nu}_t(C):=a\int_0^t(1-e^{-\beta(t-s_1)})\nu(C^{-1}(t-s_1))\beta e^{-\beta s_1}ds_1,\quad C\in\mathcal{E},$$

and $a_t := \widetilde{\nu}_t(E)$, we get $\mathbb{P}_x[X_t \in A] \geq \widetilde{\nu}_t(A) = a_t\nu_t(A)$. Recalling that $x \in E$ was arbitrary and the lower bound is independent of $x \in E$, we get ($\mathcal{B}$5a), which concludes the proof of this step.

Step 3. We show ($\mathcal{B}$5b). Using ($\mathcal{B}$5a), for any $t > 0$, we get

$$\begin{aligned}\kappa_t &:= \sup_{x,y\in E}\sup_{A\in\mathcal{E}}\left(\mathbb{P}_x[X_t \in A] - \mathbb{P}_y[X_t \in A]\right)\\ &= \sup_{x,y\in E}\sup_{A\in\mathcal{E}}\left(1 - \mathbb{P}_x[X_t \in A^c] - \mathbb{P}_y[X_t \in A]\right)\\ &\leq 1 - a_t\nu_t(A^c) - a_t\nu_t(A) = 1 - a_t < 1.\end{aligned}$$

Thus, for any $f \in \mathcal{C}_b^+(E)$ satisfying $\kappa_t e^{t\|f\|_{sp}} < 1$ for $t > 0$ small enough, using Theorem A.4.1, we get ($\mathcal{B}$5b). ♦

5.2.2 Reflected diffusions

The next two examples are related to reflected diffusions (with jumps). In a nutshell, we consider a solution to a suitable stochastic differential equation that is reflected when it reaches the boundary of the domain. Example 5.2.2 considers a compact domain reflected diffusion process studied in Menaldi and Robin (1997) and Garroni and Menaldi (2002). Example 5.2.3 could be seen as an extension of Example 5.2.2 in which jumps are allowed; see Remark 2.1b in Menaldi and Robin (2018) and references therein.

In the examples, to simplify the narrative, we use some terminology from the partial differential equations theory. In particular, we say that some function is of $\mathcal{C}^{k+\alpha}$-class (for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$) if it is k-times continuously differentiable and its kth derivative is Hölder continuous with exponent α. We refer to e.g. Section 1.1 in Garroni and Menaldi (2002) for a detailed discussion on the related terminology.

Example 5.2.2. Let $\alpha \in (0,1)$, let $\mathcal{O}$ be a bounded non-empty domain in $\mathbb{R}^d$ with $\mathcal{C}^{2+\alpha}$-class boundary $\partial\mathcal{O}$, and let E be the closure of $\mathcal{O}$. Also, let $A = (a_{ij})_{i,j=1}^d$ be a uniformly elliptic and symmetric matrix-valued mapping, with bounded and Hölder continuous (with exponent α) margins $a_{ij} : E \to \mathbb{R}$. Finally, let $b := (b_i)_{i=1}^d$ be an $\mathbb{R}^d$-valued mapping with $\mathcal{C}^{1+\alpha}(\partial\mathcal{O})$-class margins that satisfies the non-tangentiality condition, i.e. for some $b_0 > 0$ we get $b(x)\cdot n(x) \geq b_0$, $x \in \partial\mathcal{O}$, where $n(x) = (n_1(x),\ldots,n_d(x))$ denotes the unit outward normal to $\mathcal{O}$ at x.

Following arguments leading to Equation (7.1.18) in Garroni and Menaldi (2002), we get that there exists a weak solution to

$$dX_t = A^{1/2}(X_t)dW_t - b(X_t)d\xi_t, \tag{5.2.10}$$

where $A^{1/2}$ denotes the positive square root of A. Namely, there exists a pair of processes (X_t, ξ_t) with some d-dimensional Brownian motion (W_t), where the process (X_t) is understood as the reflected diffusion, and (ξ_t) describes the reflection; see Section 2.1 in Bensoussan and Lions (1984) for further discussion on the meaning of (5.2.10). From Section 7.1 in Garroni and Menaldi (2002), we get that (X_t) is a $\mathcal{C}_b$-Feller–Markov process with values in E and the transition probability satisfying

$$\mathbb{P}_x\left[X_t \in A\right] = \int_A p_t(x,y)dy, \quad t>0,\, x\in E,\, A\in\mathcal{B}(E), \tag{5.2.11}$$

where the density $(x, y) \mapsto p_t(x, y)$ is continuous for any $t > 0$. Also, using Theorem 4.3.7 from Garroni and Menaldi (2002) and the compactness of E, for any $t > 0$, we can find constants $0 < a_t \leq b_t < \infty$ such that $a_t \leq p_t(x, y) \leq b_t$, $x, y \in E$. Hence, recalling (4.1.6), we get that ($\mathcal{B}5$) is satisfied. Also, recalling that X is $\mathcal{C}_b$-Feller and E is compact, we get that X is $\mathcal{C}_0$-Feller. Consequently, using Proposition 2.1.7, we get that ($\mathcal{B}4$) is satisfied. ♦

Example 5.2.3. Using the results from the theory of integro-differential equations, one could expand Example 5.2.2 by allowing jumps. More explicitly, in Garroni and Menaldi (2002), it is shown that there exists a solution to

$$dX_t = A^{1/2}(X_t)dW_t + \int_{\mathbb{R}^d\setminus\{0\}} z\mu_X(dt,dz) - b(X_t)d\xi_t. \tag{5.2.12}$$

Here, μ_X denotes the measure associated with the Doob–Meyer decomposition of the suitable Lévy measure; we refer to Section 7.1 in Garroni and Menaldi (2002) for details. Due to the second term in (5.2.12), the process (X_t) might be seen as a reflected diffusion with jumps. Using similar logic as in Example 5.2.2, one can show that ($\mathcal{B}4$) and ($\mathcal{B}5$) are satisfied. ♦

5.3 Non-uniqueness of a solution to the optimal stopping Bellman equation

In this section, we provide computable toy examples related to the optimal stopping Bellman equation. More specifically, we provide explicit formulae for some infinite time horizon optimal stopping value functions. Also, we show the specific form of the Bellman equation with a non-unique solution. This proves that some of the results presented in Chapter 3 cannot be generalised. In particular, this applies to Theorem 3.2.6 and Theorem 3.4.3, in which we characterise the structure of solutions to the Bellman equation. In Section 5.3.1 we focus on the discrete time setting while in Section 5.3.2 we focus on the continuous time case.

5.3.1 Discrete time case

In this section, we show a dynamics with multiple solutions to the discrete time Bellman equation (3.2.4). In fact, we find explicit formulae for the value functions of the optimal stopping problems (3.2.1) and (3.2.2) and show that they are not identically equal to each other. The argument extensively uses the link between infinite and finite time horizon stopping problems stated in Theorem 3.2.6.

Example 5.3.1. Let $E := [0, +\infty) \subset \mathbb{R}$, $g \equiv c > 0$, and $G(x) := x$, $x \in E$. Let $\alpha \in [0,1]$ and (X_n), $n \in \mathbb{N}$, be a discrete time Markov process with a transition probability

$$\mathbb{P}_x[X_1 = 0] := \alpha, \, \mathbb{P}_x[X_1 = x+1] := 1-\alpha, \quad x \in E.$$

By direct calculations, we get that Assumptions ($\mathcal{A}1$), ($\mathcal{A}2'$), and ($\mathcal{A}3'$) are satisfied in this model.

Recalling (3.2.1) and (3.2.2), let us consider

$$\underline{w}(x) := \inf_{\tau \in \mathcal{T}_x^0} \ln \mathbb{E}_x \left[e^{c\tau + X_\tau} \right], \quad x \in E, \tag{5.3.1}$$

$$\overline{w}(x) := \inf_{\tau \in \mathcal{T}_{x,b}^0} \ln \mathbb{E}_x \left[e^{c\tau + X_\tau} \right], \quad x \in E. \tag{5.3.2}$$

Also, let $K := \ln \left(\frac{\alpha e^c}{1-(1-\alpha)e^c} \right)$; note that this constant is well defined if we have $(1-\alpha)e^c < 1$, otherwise we set $K := +\infty$. We show that within this model we get:

- If $\alpha \in [0, 1-e^{-c}]$, then $\underline{w}(x) = x = \overline{w}(x)$, $x \in E$.
- If $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$, then $\underline{w}(x) = x \wedge K$ and $\overline{w}(x) = x$, $x \in E$.
- If $\alpha \in (1-e^{-c-1}, 1]$, then $\underline{w}(x) = x \wedge K = \overline{w}(x)$, $x \in E$.

In particular, recalling Theorem 3.2.6, for $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$, we get two distinct solutions to the Bellman equation

$$e^{w(x)} = \min \left(e^x, e^c \left(\alpha e^{w(0)} + (1-\alpha) e^{w(x+1)} \right) \right), \quad x \in E. \tag{5.3.3}$$

Namely, we get that both $\underline{w}$ and $\overline{w}$ satisfy (5.3.3), but we get $\underline{w}(x) < \overline{w}(x)$ for $x > K$. In fact, in this example we may construct infinitely many solutions to (5.3.3); see Remark 5.3.4.

It should be noted that the equality $\underline{w}(x) = x$ corresponds to the situation when instantaneous stopping is optimal; a similar relation holds for $\overline{w}$. Thus, we get that, for α small enough (relative to c), immediate stopping is optimal. However, for a sufficiently big α, it is optimal to wait until the process goes to zero; see the argument below for details.

For transparency, we split the argument into four steps: (1) proof of $\underline{w}(x) = x$, $x \in E$, for $\alpha \in [0, 1 - e^{-c}]$; (2) proof of $\underline{w}(x) = x \wedge K$, $x \in E$, for $\alpha \in (1 - e^{-c}, 1]$; (3) proof of $\overline{w}(x) = x$, $x \in E$ for $\alpha \in [0, 1 - e^{-c-1}]$; (4) proof of $\overline{w}(x) = x \wedge K$, $x \in E$ for $\alpha \in (1 - e^{-c-1}, 1]$.

Step 1. We show that $\underline{w}(x) = x$, $x \in E$, for $\alpha \in [0, 1 - e^{-c}]$. Recalling Theorem 3.2.6, it is enough to show that $\lim_{n\to\infty} \underline{w}_n(x) = x$, $x \in E$, where the sequence $(\underline{w}_n)$ is recursively defined as

$$\begin{aligned} \underline{w}_0(x) &:= 0, \quad x \in E, \\ e^{\underline{w}_{n+1}(x)} &:= e^x \wedge e^c(\alpha e^{\underline{w}_n(0)} + (1-\alpha)e^{\underline{w}_n(x+1)}), \quad n \in \mathbb{N},\, x \in E. \end{aligned} \tag{5.3.4}$$

Recalling Proposition 3.2.4, we get $\underline{w}_n(x) \geq \underline{w}_0(x) = 0$, $n \in \mathbb{N}$, $x \in E$. Thus, noting that $e^c(\alpha e^{\underline{w}_n(0)} + (1-\alpha)e^{\underline{w}_n(1)}) \geq e^c > 1$, we get $\underline{w}_n(0) = 0$ for any $n \in \mathbb{N}$. Consequently, we get

$$e^{\underline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{\underline{w}_n(x+1)}), \quad n \in \mathbb{N},\, x \in E. \tag{5.3.5}$$

Let us now show that

$$e^{\underline{w}_n(x)} = e^x \wedge e^{c_n}, \quad n \in \mathbb{N}_*,\, x \in E, \tag{5.3.6}$$

where the sequence (c_n) is such that

$$e^{c_n} := \sum_{k=1}^{n-1} \alpha(1-\alpha)^{k-1}e^{ck} + (1-\alpha)^{n-1}e^{cn}, \quad n \in \mathbb{N}_*. \tag{5.3.7}$$

First, we we show that the sequence (c_n) is increasing. Indeed, recalling that $c > 0$, for any $n \in \mathbb{N}$, we get

$$\begin{aligned} e^{c_{n+1}} &= \sum_{k=1}^{n-1} \alpha(1-\alpha)^{k-1}e^{ck} + \alpha(1-\alpha)^{n-1}e^{cn} + (1-\alpha)^n e^{c(n+1)} \\ &= e^{c_n} - (1-\alpha)^{n-1}e^{cn} + \alpha(1-\alpha)^{n-1}e^{cn} + (1-\alpha)^n e^{c(n+1)} \\ &= e^{c_n} - (1-\alpha)^n e^{cn} + (1-\alpha)^n e^{c(n+1)} \\ &= e^{c_n} + (1-\alpha)^n e^{cn}(e^c - 1) \geq e^{c_n}. \end{aligned} \tag{5.3.8}$$

Also, noting that $\alpha \in [0, 1-e^{-c}]$ implies $(1-\alpha)e^c \geq 1$, from (5.3.7), we get that $c_n \to \infty$ as $n \to \infty$. Thus, letting $n \to \infty$ in (5.3.6) and using Theorem 3.2.6, we get $\underline{w}(x) = x$, $x \in E$.

To show (5.3.6), we proceed by induction. For $n=1$, directly from (5.3.4), we get $e^{\underline{w}_1(x)} = e^x \wedge e^c = e^x \wedge e^{c_1}$. Now, let us assume that, for some $n \in \mathbb{N}_*$, we get $e^{\underline{w}_n(x)} = e^x \wedge e^{c_n}$, $x \in E$. Then, recalling (5.3.5), for $x \in E$ such that $x+1 \geq c_n$, by direct calculation, we get

$$e^{\underline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{c_n}) = e^x \wedge e^{c_{n+1}}.$$

Now, we show the claim for $x \in E$ such that $x+1 < c_n$. To do this, let us define

$$h(y) := e^c(\alpha + (1-\alpha)e^{y+1}) - e^y, \quad y \in E. \tag{5.3.9}$$

Noting that $\alpha \in [0, 1-e^{-c}]$ implies $e^{c+1}(1-\alpha) \geq e^c(1-\alpha) \geq 1$, we get

$$\frac{d}{dy}h(y) = e^y(e^{c+1}(1-\alpha) - 1) \geq 0, \quad y \in E.$$

This, together with the estimate $h(0) = e^c\alpha + (1-\alpha)e^{c+1} - 1 \geq e^c\alpha \geq 0$, shows

$$h(y) \geq 0, \quad y \in E; \tag{5.3.10}$$

note that, in fact, this inequality is valid for any $\alpha \in [0, e^{-c-1}]$. In particular, from (5.3.10), we get $e^c(\alpha + (1-\alpha)e^{x+1}) \geq e^x$ for $x \in E$ such that $x+1 < c_n$. Thus, using (5.3.5) and the induction assumption, for $x < c_n - 1 \leq c_{n+1}$, we get

$$e^{\underline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{x+1}) = e^x = e^x \wedge e^{c_{n+1}}, \quad x \in E.$$

Consequently, we get (5.3.6). Thus, letting $n \to \infty$ and recalling that $c_n \to \infty$, we get $\underline{w}(x) = x$, $x \in E$, which concludes the proof of this step.

Step 2. We show that $\underline{w}(x) = x \wedge K$, $x \in E$, for $\alpha \in (1-e^{-c}, 1]$. As in Step 1, we show that

$$e^{\underline{w}_n(x)} = e^x \wedge e^{c_n}, \quad n \in \mathbb{N}_*, \, x \in E, \tag{5.3.11}$$

where the sequences $(\underline{w}_n)$ and (c_n) are given by (5.3.4) and (5.3.7), respectively. In this case, from $\alpha \in (1-e^{-c}, 1]$, we get $(1-\alpha)e^c < 1$. Thus, recalling (5.3.7), we get $e^{c_n} \to e^K$ as $n \to \infty$. Hence, letting $n \to \infty$ in (5.3.11), and recalling Theorem 3.2.6, we get $\underline{w}(x) = x \wedge K$, $x \in E$.

To show (5.3.11), we proceed by induction. For $n=1$, as in Step 1, we get $e^{\underline{w}_1(x)} = e^x \wedge e^c = e^x \wedge e^{c_1}$, $x \in E$. Now, let us assume that, for some $n \in \mathbb{N}_*$, we get $e^{\underline{w}_n(x)} = e^x \wedge e^{c_n}$, $x \in E$. As in Step 1, we may show that (w_n) satisfies (5.3.5). Then, for $x \in E$ such that $x+1 \geq c_n$, by direct calculation, we get

$$e^{\underline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{c_n}) = e^x \wedge e^{c_{n+1}}.$$

Now, we show the claim for $x \in E$ such that $x+1 < c_n$. Recall that, by (5.3.8), the sequence (c_n) is increasing. Thus, from (5.3.5) and the induction assumption, for $x < c_n - 1 \le c_{n+1} - 1 \le K - 1$, we get

$$e^{\underline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{x+1}). \tag{5.3.12}$$

Thus, it is enough to show $e^c(\alpha + (1-\alpha)e^{x+1}) \ge e^x$ for $x \in [0, K-1]$. Let us define $h(y) := e^c(\alpha + (1-\alpha)e^{y+1}) - e^y$, $y \in E$. Noting that

$$\frac{d}{dy} h(y) = e^y(e^{c+1}(1-\alpha) - 1),$$

we get that h is monotonic (increasing or decreasing). Thus, recalling that $e^K = \frac{\alpha e^c}{1-(1-\alpha)e^c}$, and using the estimates

$$\begin{aligned} h(0) &= e^c\alpha + e^c(1-\alpha)e - 1 \ge e^c(\alpha + (1-\alpha)) - 1 = e^c - 1 > 0, \\ h(K-1) &= e^c\alpha + e^c(1-\alpha)e^K - e^{K-1} \\ &= e^c\alpha + e^c(1-\alpha)\frac{\alpha e^c}{1-(1-\alpha)e^c} - e^{-1}\frac{\alpha e^c}{1-(1-\alpha)e^c} \\ &= \frac{\alpha e^c(1-e^{-1})}{1-(1-\alpha)e^c} > 0, \end{aligned}$$

we get that $h(y) \ge 0$ for $y \in [0, K-1]$. Thus, recalling (5.3.12), for $x \in E$ such that $x < c_n - 1 \le K - 1$, we get $e^{\underline{w}_{n+1}(x)} = e^x = e^x \wedge e^{c_{n+1}}$, which shows (5.3.11). Hence, letting $n \to \infty$ in (5.3.11) and recalling Theorem 3.2.6, we get $\underline{w}(x) = x \wedge K$, which concludes the proof of this step.

Step 3. We show that $\overline{w}(x) = x$, $x \in E$, for $\alpha \in [0, 1-e^{-c-1}]$. Recalling Theorem 3.2.6, it is enough to show $\overline{w}_n(x) = x$, $n \in \mathbb{N}$, $x \in E$, where the sequence $(\overline{w}_n)$ is recursively defined as

$$\begin{aligned} \overline{w}_0(x) &:= x, \quad x \in E, \\ e^{\overline{w}_{n+1}(x)} &:= e^x \wedge e^c(\alpha e^{\overline{w}_n(0)} + (1-\alpha)e^{\overline{w}_n(x+1)}), \quad n \in \mathbb{N}, x \in E. \end{aligned}$$

Noting that $\overline{w}_0(0) = 0$ and recalling that by Proposition 3.2.4, the map $n \mapsto \overline{w}_n(0)$ is decreasing, we get $\overline{w}_n(0) = 0$, $n \in \mathbb{N}$. Thus, we get

$$e^{\overline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{\overline{w}_n(x+1)}), \quad n \in \mathbb{N}, x \in E. \tag{5.3.13}$$

To show $\overline{w}_n(x) = x$, $n \in \mathbb{N}$, $x \in E$, we proceed by induction. The claim for $n = 0$ follows directly from the definition. Now, let us assume that, for some $n \in \mathbb{N}$, we get $\overline{w}_n(x) = x$, $x \in E$. Then, recalling (5.3.13), we get

$$e^{\overline{w}_{n+1}(x)} = e^x \wedge e^c(\alpha + (1-\alpha)e^{x+1}), \quad x \in E.$$

Also, recalling (5.3.9), (5.3.10), and the following discussion, we get

$$e^c(\alpha + (1-\alpha)e^{x+1}) \geq e^x, \quad x \in E.$$

Thus, for any $n \in \mathbb{N}$ and $x \in E$, we get $\overline{w}_n(x) = x$. Letting $n \to \infty$ and recalling Theorem 3.2.6, we get $\overline{w}(x) = x$, $x \in E$, which concludes the proof of this step.

Step 4. We show that $\overline{w}(x) = x \wedge K$, $x \in E$, for $\alpha \in (1-e^{-c-1}, 1]$. Using a discrete time version of Proposition 3.1.1, we get

$$\overline{w}(x) = \inf_{\tau \in \mathcal{T}^0} \liminf_{n\to\infty} \ln \mathbb{E}_x \left[e^{c(\tau \wedge n) + X_{\tau \wedge n}} \right], \quad x \in E. \tag{5.3.14}$$

Thus, recalling that, for $\alpha \in (1-e^{-c-1}, 1]$, we get $\underline{w}(x) = x \wedge K$ and $\underline{w}(x) \leq \overline{w}(x)$, $x \in E$, it is enough to show

$$\liminf_{n\to\infty} \mathbb{E}_x \left[e^{c(\tau_K \wedge n) + X_{\tau_K \wedge n}} \right] = e^{x \wedge K}, \quad x \in E, \tag{5.3.15}$$

where $\tau_K = \inf\{n \in \mathbb{N} \colon X_n \in [0, K]\}$. Indeed, combining (5.3.14) with (5.3.15), for any $x \in E$, we get

$$x \wedge K = \underline{w}(x) \leq \overline{w}(x) \leq \liminf_{n\to\infty} \ln \mathbb{E}_x \left[e^{c(\tau_K \wedge n) + X_{\tau_K \wedge n}} \right] = x \wedge K,$$

which proves $\overline{w}(x) = x \wedge K$, $x \in E$.

To show (5.3.15), note that, for $x \in [0, K]$, we get $\mathbb{P}_x[\tau_K = 0] = 1$, and consequently $\mathbb{E}_x \left[e^{c(\tau_K \wedge n) + X_{\tau_K \wedge n}} \right] = e^x = e^{x \wedge K}$, $n \in \mathbb{N}$. For $x > K$, we get

$$\mathbb{P}_x[\tau_K = \inf\{n \in \mathbb{N} \colon X_n = 0\}] = 1.$$

Thus, for $x > K$ and $n \in \mathbb{N}_*$, we get

$$\begin{aligned}
\mathbb{E}_x \left[e^{c(\tau_K \wedge n) + X_{\tau_K \wedge n}} \right] &= \sum_{k=1}^{n} \mathbb{E}_x \left[1_{\{\tau_K = k\}} e^{ck + X_k} \right] + \sum_{k=n+1}^{\infty} \mathbb{E}_x \left[1_{\{\tau_K = k\}} e^{cn + X_n} \right] \\
&= \sum_{k=1}^{n} \alpha(1-\alpha)^{k-1} e^{ck} + \sum_{k=n+1}^{\infty} \alpha(1-\alpha)^{k-1} e^{cn+x+n} \\
&= \sum_{k=1}^{n} \alpha(1-\alpha)^{k-1} e^{ck} + (1-\alpha)^n e^{(c+1)n} e^x.
\end{aligned} \tag{5.3.16}$$

Note that from the assumption $\alpha \in (1-e^{-c-1}, 1]$, we get $(1-\alpha)e^{c+1} < 1$, hence we get $(1-\alpha)^n e^{(c+1)n} \to 0$ as $n \to \infty$. Thus, recalling (5.3.16) and the fact that $e^K = \frac{\alpha e^c}{1-(1-\alpha)e^c}$, for $x > K$, we get

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}_x\left[e^{c(\tau_K\wedge n)+X_{\tau_K\wedge n}}\right] &\\ = \liminf_{n\to\infty}\left(\sum_{k=1}^{n}\alpha(1-\alpha)^{k-1}e^{ck} + (1-\alpha)^n e^{(c+1)n}e^x\right) &\\ = \lim_{n\to\infty}\sum_{k=1}^{n}\alpha(1-\alpha)^{k-1}e^{ck} = e^K, & \quad (5.3.17)\end{aligned}$$

which concludes the proof of (5.3.15). ♦

Remark 5.3.2. Let us now comment on the link between the uniform integrability condition from Theorem 3.2.11 and the dynamics studied in Example 5.3.1. More specifically, let $\underline{\tau} := \inf\{n \in \mathbb{N}\colon \underline{w}(X_n) = G(X_n)\}$ and let

$$Z(n) := e^{cn+X_n}, \quad n \in \mathbb{N}. \tag{5.3.18}$$

As we show now, the process $(Z(\underline{\tau}\wedge n))$, $n \in \mathbb{N}$, is $\mathbb{P}_x$-uniformly integrable if and only if $\alpha \in [0, 1-e^{-c}] \cup (1-e^{-c-1}, 1]$ or $x \leq K$. Thus, in this case, the uniform integrability of $(Z(\underline{\tau}\wedge n))$, $n \in \mathbb{N}$, is equivalent to the identity $\underline{w} \equiv \overline{w}$; cf. Theorem 3.2.11.

Using a discrete time version of Lemma 3.1.2, we get that the uniform integrability of $(Z(\underline{\tau}\wedge n))$, $n \in \mathbb{N}$, is equivalent to

$$\lim_{n\to\infty} \mathbb{E}_x[Z(\underline{\tau}\wedge n)] = \mathbb{E}_x[Z(\underline{\tau})], \quad x \in E, \tag{5.3.19}$$

provided that we get $\underline{\tau} \in \mathcal{T}_x^0$ and $\mathbb{E}_x[Z(\underline{\tau})] < \infty$. For $\alpha \in [0, 1-e^{-c}]$, we get $\underline{w}(x) = x$, $x \in E$, hence $\underline{\tau} \equiv 0$ $\mathbb{P}_x$ a.s., which directly implies (5.3.19). Next, for $\alpha \in (1-e^{-c}, 1]$, we get $\underline{w}(x) = x \wedge K$, $x \in E$, hence, for any $x \in E$, we get

$$\underline{\tau} = \inf\{n \in \mathbb{N}\colon X_n \in [0, K]\} \quad \mathbb{P}_x \text{ a.s.}$$

In particular, for $x \in [0, K]$, we get $\underline{\tau} = 0$ $\mathbb{P}_x$ a.s. and (5.3.19) holds. Also, for $x > K$, we get $\underline{\tau} = \inf\{n \in \mathbb{N}\colon X_n = 0\}$ $\mathbb{P}_x$ a.s. Then, we get $\mathbb{P}_x[\underline{\tau} = n] = \alpha(1-\alpha)^{n-1}$, $x > K$, $n \in \mathbb{N}_*$, and $\underline{\tau} \in \mathcal{T}_x^0$, $x > K$. Hence, noting that $X_{\underline{\tau}} = 0$ and using the fact that $\alpha \in (1-e^{-c}, 1]$ implies $e^c(1-\alpha) < 1$, for any $x > K$, we get

$$\mathbb{E}_x[Z(\underline{\tau})] = \mathbb{E}_x[e^{c\underline{\tau}+X_{\underline{\tau}}}] = \sum_{n=1}^{\infty} e^{cn}\alpha(1-\alpha)^{n-1} = \frac{\alpha e^c}{1-(1-\alpha)e^c} = e^K. \tag{5.3.20}$$

Also, note that from (5.3.17), for $\alpha \in (1-e^{-c-1}, 1]$ and $x > K$, we get

$$\lim_{n\to\infty} \mathbb{E}_x[Z(\underline{\tau}\wedge n)] = \lim_{n\to\infty} \mathbb{E}_x\left[e^{c(\underline{\tau}\wedge n)+X_{\underline{\tau}\wedge n}}\right] = e^K.$$

Thus, we get (5.3.19) for $\alpha \in [0, 1-e^{-c}] \cup (1-e^{-c-1}, 1]$ or $x \leq K$. Finally, we show that, for $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$ and $x > K$, we get that $(Z(\underline{\tau}\wedge n))$, $n \in \mathbb{N}$, is not $\mathbb{P}_x$-uniformly integrable. Recalling (5.3.16) and noting that $\underline{\tau} = \tau_K$ $\mathbb{P}_x$ a.s., for $x > K$, for any $n \in \mathbb{N}_*$, we get

$$\begin{aligned}\mathbb{E}_x[Z(\underline{\tau}\wedge n)] &= \mathbb{E}_x\left[e^{c(\underline{\tau}\wedge n)+X_{\underline{\tau}\wedge n}}\right]\\ &= \sum_{k=1}^{n} \alpha(1-\alpha)^{k-1}e^{ck} + (1-\alpha)^n e^{(c+1)n}e^x. \end{aligned} \tag{5.3.21}$$

Also, note that from $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$ we get $(1-\alpha)e^c < 1$ and $(1-\alpha)e^{c+1} \geq 1$. Thus, $\lim_{n\to\infty} \sum_{k=1}^{n} \alpha(1-\alpha)^{k-1}e^{ck} = \frac{\alpha e^c}{1-(1-\alpha)e^c} = e^K$ and $\lim_{n\to\infty}(1-\alpha)^n e^{(c+1)n}e^x > 1$. Consequently, recalling (5.3.20) and (5.3.21), we get

$$\mathbb{E}_x[Z(\underline{\tau})] = e^K < e^K + 1 < \lim_{n\to\infty} \mathbb{E}_x[Z(\underline{\tau}\wedge n)].$$

Thus, for $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$ and $x > K$, we get that (5.3.19) does not hold, which concludes the proof. ♢

Remark 5.3.3. Now, we show that the uniform integrability condition from Remark 5.3.2 cannot be replaced by the related condition with the stopping time corresponding to $\overline{w}$. More specifically, let $\overline{\tau} := \inf\{n \in \mathbb{N} : \overline{w}(X_n) = G(X_n)\}$ and let $(Z(n))$ be given by (5.3.18). As we show now, the process $(Z(\overline{\tau}\wedge n))$, $n \in \mathbb{N}$, is $\mathbb{P}_x$-uniformly integrable for any $\alpha \in [0, 1]$ and $x \in E$. In particular, for $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$, we get that the uniform integrability of $(Z(\overline{\tau}\wedge n))$, $n \in \mathbb{N}$, does not imply the equality of $\underline{w}$ and $\overline{w}$; cf. Remark 3.2.12.

As in Remark 5.3.2, we show that, for any $\alpha \in [0, 1]$, we get

$$\lim_{n\to\infty} \mathbb{E}_x[Z(\overline{\tau}\wedge n)] = \mathbb{E}_x[Z(\overline{\tau})]. \quad x \in E, \tag{5.3.22}$$

Note that, for $\alpha \in [0, 1-e^{-c-1}]$, we get $\overline{w}(x) = x$, $x \in E$, thus $\overline{\tau} = 0$, $\mathbb{P}_x$ a.s. and (5.3.22) holds. Also, for $\alpha \in (1-e^{-c-1}, 1]$ we get $\underline{w} \equiv \overline{w}$, hence $\underline{\tau} \equiv \overline{\tau}$. Next, recall that, by Remark 5.3.2, for $\alpha \in (1-e^{-c-1}, 1]$ and $x \in E$, we get $\lim_{n\to\infty} \mathbb{E}_x[Z(\underline{\tau}\wedge n)] = \mathbb{E}_x[Z(\underline{\tau})]$. Thus, we get (5.3.22). ♢

Remark 5.3.4. In this remark we show how to construct a solution to the Bellman equation (5.3.3), which is not identically equal to $\underline{w}$ and $\overline{w}$. We focus on the dynamics from Example 5.3.1 with $\alpha \in (1-e^{-c}, 1-e^{-c-1}]$. Let us define the map $w \in \mathcal{M}^+(E)$ by

$$w(x) := \begin{cases} x, & x \in [0,K] \cup \mathbb{N}, \\ K, & \text{otherwise.} \end{cases}$$

We show that w is a solution to the Bellman equation (5.3.3). Indeed, noting that $w(x) = \overline{w}(x)$, $x \in \mathbb{N}$, where $\overline{w}$ is given by (5.3.2), and recalling that $\overline{w}$ is a solution to (5.3.3), we get

$$\begin{aligned} e^{w(x)} = e^{\overline{w}(x)} &= e^x \wedge e^c \left(\alpha e^{\overline{w}(0)} + (1-\alpha)e^{\overline{w}(x+1)}\right) \\ &= e^x \wedge e^c \left(\alpha e^{w(0)} + (1-\alpha)e^{w(x+1)}\right), \quad x \in \mathbb{N}. \end{aligned}$$

Similarly, noting that $w(x) = \underline{w}(x)$, $x \in E \setminus \mathbb{N}$, where $\underline{w}$ is given by (5.3.1), we get

$$\begin{aligned} e^{w(x)} = e^{\underline{w}(x)} &= e^x \wedge e^c \left(\alpha e^{\underline{w}(0)} + (1-\alpha)e^{\underline{w}(x+1)}\right) \\ &= e^x \wedge e^c \left(\alpha e^{w(0)} + (1-\alpha)e^{w(x+1)}\right), \quad x \in E \setminus \mathbb{N}. \end{aligned}$$

Hence, w is a solution to (5.3.3), which is different from $\underline{w}$ and $\overline{w}$; cf. Theorem 3.2.6. Also, note that w is discontinuous. In fact, using similar logic, we may construct infinitely many (discontinuous) solutions to (5.3.3). $\Diamond$

5.3.2 Continuous time case

In this section, we give an example for the non-uniqueness of a solution to the continuous time Bellman equation (3.4.3). The example takes the form of a piecewise constant Markov process, where the jump measure is linked to the dynamics studied in Example 5.3.1. Using a suitable identification, we show that the corresponding Bellman equation admits multiple solutions.

Example 5.3.5. In this example we use the dynamics from Example 5.3.1 to get a piecewise deterministic (piecewise constant) continuous time Markov process X with values in the state space $E := [0, +\infty)$. The process construction follows the logic of Example 5.2.1, thus we provide only an outline.

First, let (Y_n) be a discrete time Markov process with the dynamics studied in Example 5.3.1, i.e.

$$\mathbb{P}_x[Y_1 = 0] = \alpha,\ \mathbb{P}_x[Y_1 = x+1] = 1-\alpha, \quad x \in E, \tag{5.3.23}$$

for some $\alpha \in [0,1]$. Also, let $(T_n)_{n=1}^{\infty}$ be an increasing sequence of non-negative random variables. We set $T_0 \equiv 0$ and assume that the increments $(T_{n+1} - T_n)$, $n \in \mathbb{N}$, are exponentially distributed with the rate parameter $\beta > 0$. Also, we assume that the jump times (T_n) are independent of (Y_n). Finally, we define the process X as $X_t := Y_n$ for $t \in [T_n, T_{n+1})$. We refer to Davis (1993) for a more detailed discussion on the process construction.

By analogy to Example 5.3.1, we set $g \equiv C$ with $C \in (0, \beta)$ and $G(x) = x$, $x \in E$. Also, we consider the continuous time optimal stopping problem

$$\underline{w}(x) := \inf_{\tau \in \mathcal{T}_x} \ln \mathbb{E}_x \left[e^{C\tau + X_\tau} \right], \quad x \in E. \tag{5.3.24}$$

Following (3.4.3), with this problem we associate the continuous time Bellman equation

$$e^{w(x)} = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{C(\tau \wedge t) + 1_{\{\tau < t\}} X_\tau + 1_{\{\tau \geq t\}} w(X_t)} \right], \quad w \in \mathcal{M}^+(E),\, x \in E,\, t \geq 0. \tag{5.3.25}$$

We show that this equation admits multiple solutions. Before we present a detailed argument, let us explain the intuition. Since $C > 0$, it is optimal to stop the process in (5.3.24) only at the times when the process (X_t) is subject to a jump. Thus, (5.3.24), at least formally, should be equivalent to

$$\underline{w}(x) = \inf_{\nu \in \mathcal{T}_x^0} \ln \mathbb{E}_x [e^{CT_\nu + X_{T_\nu}}] = \inf_{\nu \in \mathcal{T}_x^0} \ln \mathbb{E}_x [e^{CT_\nu + Y_\nu}], \quad x \in E.$$

After a suitable embedding, this could be seen as a discrete time optimal stopping problem with the corresponding Bellman equation of the form

$$e^{w(x)} = \min \left(e^x, \mathbb{E}_x \left[e^{CT_1 + w(Y_1)} \right] \right), \quad w \in \mathcal{M}^+(E),\, x \in E. \tag{5.3.26}$$

Using the independence of (Y_n) and (T_n), and the fact that T_1 is exponentially distributed, for any $w \in \mathcal{M}^+(E)$ and $x \in E$, we get

$$\begin{aligned}
\mathbb{E}_x \left[e^{CT_1 + w(Y_1)} \right] &= \int_0^\infty \beta e^{-t(\beta - C)} dt \left(\alpha e^{w(0)} + (1-\alpha) e^{w(x+1)} \right) \\
&= \frac{\beta}{\beta - C} \left(\alpha e^{w(0)} + (1-\alpha) e^{w(x+1)} \right).
\end{aligned}$$

Thus, setting $c := \ln \left(\frac{\beta}{\beta - C} \right)$, we get that (5.3.26) could be rewritten as

$$e^{w(x)} = \min \left(e^x, e^c \left(\alpha e^{w(0)} + (1-\alpha) e^{w(x+1)} \right) \right), \quad w \in \mathcal{M}^+(E),\, x \in E.$$

Since, by Example 5.3.1, this equation admits multiple solutions, we should also get multiple solutions to (5.3.25).

Now, we provide a more detailed argument. As formalisation of the preceding intuition turned out to be rather cumbersome, we use a more direct method. First, we show that Assumptions ($\mathcal{A}1$)–($\mathcal{A}4$) are satisfied in this model. Also, we set $c := \ln\left(\frac{\beta}{\beta - C}\right)$ and show that, for $\alpha \in [0, 1 - e^{-c-1}]$, the map $w(x) := x$, $x \in E$, is a solution to (5.3.25). Next, we show that, for $\alpha \in (1 - e^{-c}, 1]$, we get $\underline{w}(x) \leq x \wedge K$, $x \in E$, where $\underline{w}$ is given by (5.3.24) and $K := \ln\left(\frac{\alpha e^c}{1-(1-\alpha)e^c}\right)$. Thus, recalling that, by Theorem 3.4.3, the map $\underline{w}$ is a solution to (5.3.25), we get multiple solutions to the continuous time Bellman equation. More specifically, we get that, for $\alpha \in (1 - e^{-c}, 1 - e^{-c-1}]$, both $\underline{w}$ and w are solutions to (5.3.25), but for $x > K$ we get $\underline{w}(x) \leq K < x = w(x)$.

For transparency, we split the rest of the argument into three steps: (1) proof that Assumptions ($\mathcal{A}1$)–($\mathcal{A}4$) are satisfied; (2) proof that $w(x) := x$, $x \in E$, is a solution to (5.3.25) for $\alpha \in [0, 1-e^{-c-1}]$; (3) proof that $\underline{w}(x) \leq x \wedge K$, $x \in E$, for $\alpha \in (1 - e^{-c}, 1]$.

Step 1. We show that Assumptions ($\mathcal{A}1$)–($\mathcal{A}4$) are satisfied in this model. First, using Theorem 25.5 and Theorem 27.6 from Davis (1993), as in Example 5.2.1, we get that $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a continuous time standard $\mathcal{C}_b$-Feller-Markov process with values in $(E, \mathcal{E})$. Second, we note that Assumption ($\mathcal{A}1$) follows directly from the fact that $g(x) = C > 0$, $x \in E$ and $G(x) = x$, $x \in E$. Third, we use Lemma 5.1.1 to show that Assumptions ($\mathcal{A}2$)–($\mathcal{A}3$) are satisfied. Let $T \geq 0$, $\zeta_T := \sup_{t\in[0,T]} e^{X_t}$, and $K \subset E$ be a compact set. We show that

$$\lim_{m\to\infty} \sup_{x\in K} \mathbb{E}_x \left[\zeta_T 1_{\{\zeta_T \geq e^m\}}\right] = 0, \tag{5.3.27}$$

which, combined with Lemma 5.1.1, implies ($\mathcal{A}2$)–($\mathcal{A}3$). Note that, for any $x \in E$ and $n \in \mathbb{N}$, on the event $\{T \in [T_n, T_{n+1})\}$, we get $\zeta_T \leq e^{x+n}$ $\mathbb{P}_x$ a.s. Thus, setting $L_K := \sup_{x\in K} |x|$, for any $m \in \mathbb{N}$, we get

$$\begin{aligned}
\sup_{x\in K} \mathbb{E}_x \left[\zeta_T 1_{\{\zeta_T \geq e^m\}}\right] &= \sup_{x\in K} \mathbb{E}_x \left[\sum_{n=0}^{\infty} 1_{\{T\in[T_n,T_{n+1})\}} \zeta_T 1_{\{\zeta_T \geq e^m\}}\right] \\
&\leq \sup_{x\in K} \mathbb{E}_x \left[\sum_{n=0}^{\infty} 1_{\{T\in[T_n,T_{n+1})\}} e^{x+n} 1_{\{e^{x+n}\geq e^m\}}\right] \\
&\leq \sup_{x\in K} \sum_{n=0}^{\infty} \mathbb{P}_x \left[T \in [T_n, T_{n+1})\right] e^{L_K+n} 1_{\{n\geq m-L_K\}}.
\end{aligned} \tag{5.3.28}$$

Next, recalling that $(T_{n+1} - T_n)$, $n \in \mathbb{N}$, follows the exponential distribution with the rate parameter β, we get that T_n, $n \in \mathbb{N}_*$, follows the Erlang distribution with the shape parameter n and the rate parameter β. Hence, for any $x \in E$, $n \in \mathbb{N}_*$, and $s \geq 0$, we get

$$\begin{aligned}
\mathbb{P}_x[T_n \leq s < T_{n+1}] &= \mathbb{P}_x[T_n \leq s, s - T_n < T_{n+1} - T_n] \\
&= \int_0^s \left(\int_{s-t_1}^{\infty} \beta e^{-\beta t_2} dt_2 \right) \frac{\beta^n t_1^{n-1} e^{-\beta t_1}}{(n-1)!} dt_1 \\
&= \int_0^s e^{-\beta(s-t_1)} \frac{\beta^n t_1^{n-1} e^{-\beta t_1}}{(n-1)!} dt_1 = e^{-\beta s} \frac{(\beta s)^n}{n!}; \qquad (5.3.29)
\end{aligned}$$

also, recalling that $T_0 \equiv 0$, we get that this formula is valid for $n = 0$ as well. Thus, recalling (5.3.28) and denoting by $\lfloor a \rfloor := \sup\{k \in \mathbb{Z} \colon k \leq a\}$ the integer part of $a \in \mathbb{R}$, for $m \geq L_K$, we get

$$\begin{aligned}
\sup_{x \in K} \mathbb{E}_x \left[\zeta_T 1_{\{\zeta_T \geq e^m\}} \right] &\leq \sum_{n=\lfloor m - L_k \rfloor}^{\infty} e^{-\beta T} \frac{(\beta T)^n}{n!} e^{L_K + n} \\
&= e^{L_K - \beta T} \sum_{n=\lfloor m - L_K \rfloor}^{\infty} \frac{(\beta T e)^n}{n!}.
\end{aligned}$$

Noting that $\sum_{n=0}^{\infty} \frac{(\beta T e)^n}{n!} = e^{\beta T e} < \infty$, we get $\lim_{m \to \infty} \sum_{n=\lfloor m - L_K \rfloor}^{\infty} \frac{(\beta T e)^n}{n!} = 0$, which shows (5.3.27). Thus, recalling Lemma 5.1.1, we get that Assumptions ($\mathcal{A}2$)–($\mathcal{A}3$) are satisfied. Finally, exactly as in Example 5.2.1, we show that Assumption ($\mathcal{A}4$) is satisfied; see Step 1 in the argument in Example 5.2.1 for details. This concludes the proof of this part.

Step 2. W show that, for $\alpha \in [0, 1 - e^{-c-1}]$, the map $w(x) := x$, $x \in E$, is a solution to (5.3.25). Note that it is enough to show that, for any $x \in E$, the process $Z(t) := e^{Ct + X_t}$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale. Indeed, from the submartingale property, using Doob's optional stopping theorem, for $x \in E$, $t \geq 0$, and $\tau \in \mathcal{T}$, we get $e^x \leq \mathbb{E}_x \left[e^{C(\tau \wedge t) + X_{\tau \wedge t}} \right]$. Also, noting that, for any $x \in E$, the process $Z(0 \wedge t) = e^{X_0}$, $t \geq 0$, is a $\mathbb{P}_x$-martingale, we get

$$\begin{aligned}
e^{w(x)} = e^x &= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{C(\tau \wedge t) + X_{\tau \wedge t}} \right] \\
&= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{C(\tau \wedge t) + 1_{\{\tau < t\}} X_\tau + 1_{\{\tau \geq t\}} w(X_t)} \right], \quad x \in E, \, t \geq 0,
\end{aligned}$$

which shows that $w(x) = x$, $x \in E$, is a solution to (5.3.25).

Let us now show that, for any $x \in E$, the process $Z(t) := e^{Ct+X_t}$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale. Fix $t, s \geq 0$ and note that, using the Markov property, we get

$$\mathbb{E}_x\left[e^{C(t+s)+X_{t+s}}|\mathcal{F}_t\right] = e^{C(t+s)}\mathbb{E}_x[e^{X_{t+s}}|\mathcal{F}_t] = e^{Ct}e^{Cs}\mathbb{E}_{X_t}[e^{X_s}], \quad x \in E.$$

Thus, to conclude the proof of the submartingale property, it is enough to show

$$e^{Cs}\mathbb{E}_x[e^{X_s}] \geq e^x, \quad x \in E. \tag{5.3.30}$$

To show this, note that, using the independence of (T_n) and (Y_n), for any $x \in E$, we get

$$\begin{aligned}\mathbb{E}_x[e^{X_s}] &= \sum_{n=0}^{\infty}\mathbb{E}_x[1_{\{T_n \leq s < T_{n+1}\}}e^{X_s}] \\ &= \sum_{n=0}^{\infty}\mathbb{E}_x[1_{\{T_n \leq s < T_{n+1}\}}e^{Y_n}] = \sum_{n=0}^{\infty}\mathbb{P}[T_n \leq s < T_{n+1}]\mathbb{E}_x[e^{Y_n}]. \end{aligned}\tag{5.3.31}$$

Now, we show that, for any $n \in \mathbb{N}_*$, we get

$$\mathbb{E}_x[e^{Y_n}] = \sum_{k=0}^{n-1}\alpha(1-\alpha)^k e^k + (1-\alpha)^n e^{x+n}, \quad x \in E. \tag{5.3.32}$$

For $n = 1$, directly from (5.3.23), we get $\mathbb{E}_x[e^{Y_1}] = \alpha e^0 + (1-\alpha)e^{x+1}$, $x \in E$, and (5.3.32) holds. Let us assume that the claim holds for some $n \in \mathbb{N}_*$. Then, using the Markov property, we get

$$\begin{aligned}\mathbb{E}_x\left[e^{Y_{n+1}}\right] &= \mathbb{E}_x\left[\mathbb{E}_{Y_1}\left[e^{Y_n}\right]\right] \\ &= \mathbb{E}_x\left[\sum_{k=0}^{n-1}\alpha(1-\alpha)^k e^k + (1-\alpha)^n e^{Y_1+n}\right] \\ &= \sum_{k=0}^{n-1}\alpha(1-\alpha)^k e^k + \alpha(1-\alpha)^n e^n + (1-\alpha)^{n+1}e^{x+n+1}, \quad x \in E,\end{aligned}$$

which shows (5.3.32). Thus, combining (5.3.31) with (5.3.29) and (5.3.32), and noting that $\mathbb{E}_x[e^{Y_0}] = e^x$, $x \in E$, for any $x \in E$, we get

$$\mathbb{E}_x[e^{X_s}] = e^{-\beta s}e^x + \sum_{n=1}^{\infty}e^{-\beta s}\frac{(\beta s)^n}{n!}\left(\sum_{k=0}^{n-1}\alpha(1-\alpha)^k e^k + (1-\alpha)^n e^{x+n}\right). \tag{5.3.33}$$

Now, we consider two cases and use separate arguments for $\alpha = 1 - e^{-1}$ and for $\alpha \in [0, 1 - e^{-c-1}] \setminus \{1 - e^{-1}\}$. If $\alpha = 1 - e^{-1}$, then $(1-\alpha)e = 1$, and we get

$$\begin{aligned}
\mathbb{E}_x[e^{X_s}] &= e^{-\beta s} e^x + \sum_{n=1}^{\infty} e^{-\beta s} \frac{(\beta s)^n}{n!} (n\alpha + e^x) \\
&= \alpha\beta s e^{-\beta s} \sum_{n=0}^{\infty} \frac{(\beta s)^n}{n!} + e^x e^{-\beta s} \sum_{n=0}^{\infty} \frac{(\beta s)^n}{n!} \\
&= \alpha\beta s + e^x, \quad x \in E.
\end{aligned}$$

Thus, $e^{Cs}\mathbb{E}_x[e^{X_s}] \geq e^x$, $x \in E$, which shows (5.3.30) for $\alpha = 1 - e^{-1}$. Let us now show that, for $\alpha \in [0, 1 - e^{-c-1}] \setminus \{1 - e^{-1}\}$, we also get (5.3.30). Recalling (5.3.33), for any $x \in E$, we get

$$\begin{aligned}
\mathbb{E}_x[e^{X_s}] &= e^{-\beta s} e^x + \sum_{n=1}^{\infty} e^{-\beta s} \frac{(\beta s)^n}{n!} \left(\alpha \frac{1 - (1-\alpha)^n e^n}{1 - (1-\alpha)e} + (1-\alpha)^n e^{x+n} \right) \\
&= \frac{\alpha e^{-\beta s}}{1 - (1-\alpha)e} \left(\sum_{n=1}^{\infty} \frac{(\beta s)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\beta s(1-\alpha)e)^n}{n!} \right) \\
&\quad + e^{-\beta s} e^x \sum_{n=0}^{\infty} \frac{(\beta s(1-\alpha)e)^n}{n!} \\
&= \frac{\alpha e^{-\beta s}}{1 - (1-\alpha)e} \left(e^{\beta s} - e^{\beta s(1-\alpha)e} \right) + e^x e^{\beta s((1-\alpha)e-1)}.
\end{aligned} \tag{5.3.34}$$

Also, noting that the condition $\alpha \in [0, 1-e^{-c-1}] \setminus \{1-e^{-1}\}$ implies $(1-\alpha)e \neq 1$, we get

$$\frac{\alpha e^{-\beta s}}{1 - (1-\alpha)e} \left(e^{\beta s} - e^{\beta s(1-\alpha)e} \right) \geq 0. \tag{5.3.35}$$

Next, recalling that $e^c = \frac{\beta}{\beta - C}$, from the assumption $\alpha \leq 1 - e^{-c-1}$, we get $\alpha \leq 1 - e^{-1} + e^{-1}\frac{C}{\beta}$, which shows $(1-\alpha)e \geq 1 - \frac{C}{\beta}$. Thus, combining (5.3.34) with (5.3.35), we get

$$e^{Cs}\mathbb{E}_x[e^{X_s}] \geq e^{Cs} e^x e^{\beta s((1-\alpha)e-1)} \geq e^{Cs} e^x e^{\beta s(-\frac{C}{\beta})} = e^x,$$

which concludes the proof of (5.3.30).

Step 3. We show that $\underline{w}(x) \leq x \wedge K$, $x \in E$, for $\alpha \in (1 - e^{-c}, 1]$, where $\underline{w}$ is given by (5.3.24) and $K := \ln\left(\frac{\alpha e^c}{1-(1-\alpha)e^c}\right)$. Note that it is enough to show

$$\ln \mathbb{E}_x\left[e^{C\tau_K + X_{\tau_K}}\right] = x \wedge K, \quad x \in E, \tag{5.3.36}$$

where $\tau_K := \inf\{t \geq 0\colon X_t \in [0, K]\}$. Note that, for $x \in [0, K]$, we get $\tau_K \equiv 0$ $\mathbb{P}_x$ a.s., which directly implies (5.3.36). Also, for $x > K$, we get $\mathbb{P}_x\left[\tau_K = \inf\{t \geq 0\colon X_t = 0\}\right] = 1$. Thus, noting that $X_{\tau_K} = 0$, for $x > K$, we get

$$\begin{aligned}
\mathbb{E}_x\left[e^{C\tau_K + X_{\tau_K}}\right] &= \sum_{n=1}^{\infty} \mathbb{E}_x\left[e^{CT_n} 1_{\{\tau_K = T_n\}}\right] \\
&= \sum_{n=1}^{\infty} \mathbb{E}_x\left[e^{CT_n} 1_{\{Y_1 = x+1, Y_2 = x+2, \ldots, Y_{n-1} = x+n-1, Y_n = 0\}}\right] \\
&= \sum_{n=1}^{\infty} \mathbb{E}_x\left[e^{CT_n}\right](1-\alpha)^{n-1}\alpha. \qquad (5.3.37)
\end{aligned}$$

Next, recall that T_n, $n \in \mathbb{N}_*$, follows the Erlang distribution with the shape parameter n and the rate parameter β, thus its moment generating function is given by $M_{T_n}(t) := \mathbb{E}[e^{tT_n}] = \left(1 - \frac{t}{\beta}\right)^{-n}$, $t < \beta$. Also, recalling that $e^c = \frac{\beta}{\beta - C}$, noting that $e^K = \frac{\alpha e^c}{1-(1-\alpha)e^c} = \frac{\alpha\beta}{\alpha\beta - C}$, and using (5.3.37), for $x > K$, we get

$$\mathbb{E}_x\left[e^{C\tau_K + X_{\tau_K}}\right] = \sum_{n=1}^{\infty}\left(1 - \frac{C}{\beta}\right)^{-n}(1-\alpha)^{n-1}\alpha = \frac{\alpha\beta}{\alpha\beta - C} = e^K,$$

where, to find the infinite sum, we used the fact that $\alpha > 1 - e^{-c} = \frac{C}{\beta}$ implies $\frac{1-\alpha}{1-\frac{C}{\beta}} < 1$. Thus, recalling (5.3.36), we get $\underline{w}(x) \leq x \wedge K$, $x \in E$, for $\alpha \in (1 - e^{-c}, 1]$, which concludes the proof. ♦

Appendix A

Auxiliary results

In this chapter, we discuss some auxiliary results used in the thesis. In particular, in Section A.1, we present some properties of the discrete time optimal stopping Bellman operator; this is used in Section 3.2. Next, in Section A.2, we review the properties of the Snell envelope, which facilitate the analysis in Section 3.3. Also, in Section A.3, we discuss long-run dyadic impulse control problems; this is used in Section 4.3. Finally, in Section A.4, we collect some properties of the Multiplicative Poisson Equation and the associated change of measure transformation. This is extensively used in Section 4.3.

Some of the results presented in this appendix are relatively standard and are derived from the literature. In this case, instead of providing proofs, we give the specific references. However, sometimes we need to adjust the results to our setting, and in this case we provide more detailed arguments.

A.1 Discrete time Bellman operator

In this section, we present some additional properties of the discrete time optimal stopping Bellman operator given by (3.2.6). In particular, we show that the iterates of this operator could be associated with a suitable optimal stopping problem. This is used in the proof of Proposition 3.2.4.

In this section $((X_n)_{n\in\mathbb{N}}, (\mathbb{P}_x)_{x\in E})$ is a discrete time standard Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$; note that here we do not need the Feller property. Let $g \in \mathcal{C}_b(E)$, $G \in \mathcal{C}^+(E)$, and assume ($\mathcal{A}2'$). Note that we consider a generic $g \in \mathcal{C}_b(E)$. i.e. we do not need the non-negativity condition; cf. Assumption ($\mathcal{A}1$). Also, Assumption ($\mathcal{A}3'$) is not used in this section.

Recall the Bellman operator S given by (3.2.6). In the following, we fix some map $h \in \mathcal{C}^+(E)$ satisfying $0 \leq h(x) \leq G(x)$, $x \in E$, and recursively define the sequence of functions $(w_n)_{n\in\mathbb{N}}$ given by

$$\begin{aligned} w_0(x) &:= h(x), \quad x \in E, \\ w_{n+1}(x) &:= \ln Se^{w_n}(x), \quad n \in \mathbb{N},\, x \in E. \end{aligned} \tag{A.1.1}$$

Note that setting $h \equiv 0$, we recover the sequence $(\underline{w}_n)$ given by (3.2.10). Similarly, for $h \equiv G$, we get that w_n coincides with $\overline{w}_n$ given by (3.2.11).

In Proposition A.1.1 we give a probabilistic interpretation of the sequence (w_n). This is a relatively standard result; see e.g. Section 2.2 in Shiryaev (1978). However, in the literature, usually it is assumed that $g \equiv 0$ and $h \equiv G$. Thus, for completeness, we provide a more detailed argument.

Proposition A.1.1. *Let the sequence (w_n) be given by (A.1.1). Then, we get*

$$w_n(x) = \inf_{\tau \le n} \ln \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + 1_{\{\tau<n\}} G(X_\tau) + 1_{\{\tau=n\}} h(X_n)} \right], \quad n \in \mathbb{N},\, x \in E. \tag{A.1.2}$$

Also, for any $n \in \mathbb{N}$ and $x \in E$, the stopping time

$$\tau_n := \inf \{k \in \mathbb{N} \colon w_{n-k}(X_k) = G(X_k)\} \wedge n$$

is optimal for $w_n(x)$. Moreover, for any $n \in \mathbb{N}$ and $x \in E$, the process

$$z_n(k) := e^{\sum_{i=0}^{k-1} g(X_i) + w_{n-k}(X_k)}, \quad k = 0, \ldots, n$$

is a $\mathbb{P}_x$-submartingale and $(z_n(\tau_n \wedge k))$, $k = 0, \ldots, n$, is a $\mathbb{P}_x$-martingale.

Proof. Let $n \in \mathbb{N}$ be fixed. First, we show the submartingale property of the process $(z_n(k))$, $k = 0, \ldots, n$. Directly from the fact $e^{w_{n-k}(x)} = Se^{w_{n-k-1}}(x)$, $k = 0, \ldots, n-1$, $x \in E$, we get $e^{g(x)}\mathbb{E}_x \left[e^{w_{n-k-1}(X_1)} \right] \ge e^{w_{n-k}(x)}$, $x \in E$, $k = 0, \ldots, n-1$. Thus, using the Markov property, for any $x \in E$ and $k = 0, \ldots, n-1$, we get

$$\begin{aligned} \mathbb{E}_x \left[z_n(k+1) | \mathcal{F}_k \right] &= e^{\sum_{i=0}^{k-1} g(X_i)} e^{g(X_k)} \mathbb{E}_{X_k} \left[e^{w_{n-k-1}(X_1)} \right] \\ &\ge e^{\sum_{i=0}^{k-1} g(X_i)} e^{w_{n-k}(X_k)} = z_n(k), \end{aligned}$$

which shows that $(z_n(k))$, $k = 0, \ldots, n$, is a $\mathbb{P}_x$-submartingale.

Second, we show the martingale property of the process $(z_n(\tau_n \wedge k))$, $k = 0, \ldots, n$. Note that, from (A.1.1), for any $k = 0, \ldots, n-1$, on the event $\{\tau_n > k\}$, we get $w_{n-k}(X_k) < G(X_k)$ and, consequently, we get

$$e^{w_{n-k}(X_k)} = e^{g(X_k)} \mathbb{E}_{X_k} \left[e^{w_{n-k-1}(X_1)} \right].$$

Thus, for any $x \in E$ and $k = 0, \ldots, n-1$, we get

$$\begin{aligned}
&\mathbb{E}_x[z_n(\tau_n \wedge (k+1))|\mathcal{F}_k] \\
&\qquad = 1_{\{\tau_n \le k\}} z_n(\tau_n) + 1_{\{\tau_n > k\}} e^{\sum_{i=0}^{k} g(X_i)} \mathbb{E}_x \left[e^{w_{n-k-1}(X_{k+1})} | \mathcal{F}_k \right] \\
&\qquad = 1_{\{\tau_n \le k\}} z_n(\tau_n) + 1_{\{\tau_n > k\}} e^{\sum_{i=0}^{k-1} g(X_i)} e^{g(X_k)} \mathbb{E}_{X_k} \left[e^{w_{n-k-1}(X_1)} \right] \\
&\qquad = 1_{\{\tau_n \le k\}} z_n(\tau_n \wedge k) + 1_{\{\tau_n > k\}} e^{\sum_{i=0}^{\tau_n \wedge k-1} g(X_i)} e^{w_{n-k}(X_{\tau_n \wedge k})} \\
&\qquad = z_n(\tau_n \wedge k),
\end{aligned}$$

which shows that $(z_n(k \wedge \tau_n))$, $k = 0, \ldots, n$, is a $\mathbb{P}_x$-martingale.

Finally, we show (A.1.2) and the optimality of τ_n. Using the submartingale property of (z_n), Doob's optional stopping theorem, and the facts that $w_{n-k}(\cdot) \le G(\cdot)$, $k = 0, \ldots, n$, and $w_0 \equiv h$, for any stopping time $\tau \le n$, we get

$$\begin{aligned}
e^{w_n(x)} = \mathbb{E}_x\left[z_n(0)\right] &\le \mathbb{E}_x\left[z_n(\tau)\right] \\
&= \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + 1_{\{\tau < n\}} w_{n-\tau}(X_\tau) + 1_{\{\tau = n\}} w_0(X_n)} \right] \\
&\le \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + 1_{\{\tau < n\}} G(X_\tau) + 1_{\{\tau = n\}} h(X_n)} \right], \quad x \in E.
\end{aligned}$$

Consequently, we get

$$e^{w_n(x)} \le \inf_{\tau \le n} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + 1_{\{\tau < n\}} G(X_\tau) + 1_{\{\tau = n\}} h(X_n)} \right], \quad x \in E. \tag{A.1.3}$$

Also, using the martingale property of $(z_n(\tau_n \wedge k))$ and the fact that, on the event $\{\tau_n < n\}$, we get $w_{n-\tau_n}(X_{\tau_n}) = G(X_{\tau_n})$, we also get

$$\begin{aligned}
e^{w_n(x)} &= \mathbb{E}_x\left[z_n(n \wedge \tau_n)\right] \\
&= \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau_n-1} g(X_i) + 1_{\{\tau_n < n\}} w_{n-\tau_n}(X_{\tau_n}) + 1_{\{\tau_n = n\}} w_0(X_n)} \right] \\
&= \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau_n-1} g(X_i) + 1_{\{\tau_n < n\}} G(X_{\tau_n}) + 1_{\{\tau_n = n\}} h(X_n)} \right], \quad x \in E.
\end{aligned}$$

Thus, recalling (A.1.3), we get

$$\begin{aligned}
e^{w_n(x)} &= \inf_{\tau \le n} \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau-1} g(X_i) + 1_{\{\tau < n\}} G(X_\tau) + 1_{\{\tau = n\}} h(X_n)} \right] \\
&= \mathbb{E}_x \left[e^{\sum_{i=0}^{\tau_n-1} g(X_i) + 1_{\{\tau_n < n\}} G(X_{\tau_n}) + 1_{\{\tau_n = n\}} h(X_n)} \right], \quad x \in E,
\end{aligned}$$

which concludes the proof. □

A.2 Snell envelope

In this section, we present some auxiliary results related to the Snell envelopes corresponding to continuous time optimal stopping problems. In particular, we show a martingale characterisation of the value process and provide formulae for ε-optimal and optimal stopping times. This is used in the proof of Theorem 3.3.2. Some of the results presented in this section are based on Fakeev (1970, 1971); see also El Karoui (1981) and Appendix D in Karatzas and Shreve (1998*b*) for a similar discussion.

We start with selected basic properties of the Snell envelope. To simplify the narrative, we state them in a generic, possibly non-Markovian setting. More specifically, let $(Z(t))$, $t \geq 0$, be a right-continuous stochastic process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, which satisfies usual conditions. We assume that Z satisfies

$$\mathbb{E}\left[\sup_{t\geq 0} |Z(t)|\right] < \infty. \tag{A.2.1}$$

We define the Snell envelope of Z by

$$z(t) := \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}\left[Z(\tau)|\mathcal{F}_t\right], \quad t \geq 0, \tag{A.2.2}$$

where, for any $t \geq 0$, the essential infimum is taken over all $\mathbb{P}$-a.s. finite stopping times τ satisfying $\tau \geq t$. Typically, the Snell envelope is the largest submartingale dominated by the process $(Z(t))$, $t \geq 0$. Also, it facilitates the analysis of optimal and ε-optimal stopping times for the optimal stopping problem associated with $(Z(t))$, $t \geq 0$.

In Theorem A.2.1 we collect the properties of $(z(t))$, $t \geq 0$, which are used in this thesis. For the proof, see e.g. Theorem 2 and Theorem 4 in Fakeev (1970) or Proposition D.2 and Theorem D.7 in Karatzas and Shreve (1998*b*).

Theorem A.2.1. *Let the process $(z(t))$, $t \geq 0$, be given by (A.2.2). Then:*

(1) *The process $(z(t))$, $t \geq 0$, is a submartingale.*

(2) *For any $\varepsilon > 0$, the stopping time*

$$\tau_\varepsilon := \inf\{t \geq 0 : z(t) \geq -\varepsilon + Z(t)\}$$

is ε-optimal for $\inf_\tau \mathbb{E}\left[Z(\tau)1_{\{\tau<\infty\}} + \liminf_{T\to\infty} Z(T)1_{\{\tau=\infty\}}\right]$.

(3) *We get* $\mathbb{E}[z(0)] = \inf_\tau \mathbb{E}\left[Z(\tau)1_{\{\tau<\infty\}} + \liminf_{T\to\infty} Z(T)1_{\{\tau=\infty\}}\right]$.

Remark A.2.2. Note that the specific form of the optimal stopping problem in Theorem A.2.1 is aligned with (3.1.6). More specifically, the term $\liminf_{T\to\infty} Z(T)1_{\{\tau=\infty\}}$ allows for a possible stopping at infinity; see e.g. Equation (2) in Fakeev (1970) for a further discussion. $\Diamond$

Remark A.2.3. Theorem A.2.1 gives a characterisation of ε-optimal stopping times for the stopping problem related to Z. Under the additional continuity assumption, it is also possible to get an optimal stopping time; see point (b) of Theorem 4 in Fakeev (1970). $\Diamond$

The next lemma provides an efficient procedure for finding the Snell envelope. For the proof, see the unnumbered lemma in Fakeev (1971). Note that in Lemma A.2.4, to simplify the notation, we use $\mathbb{Q} \subset \mathbb{R}$ to denote the set of rational numbers. Its countability guarantees that the maps defined in (A.2.3) are measurable.

Lemma A.2.4. *Let the sequence* $(h_n)_{n\in\mathbb{N}}$ *be given recursively by*

$$\begin{aligned} h_0(t) &:= Z(t), \quad t \geq 0, \\ h_{n+1}(t) &:= \inf_{\substack{s\geq t \\ s\in\mathbb{Q}}} \mathbb{E}\left[h_n(s)|\mathcal{F}_t\right], \quad n \in \mathbb{N},\, t \geq 0. \end{aligned} \tag{A.2.3}$$

Also, let $(z(t))$, $t \geq 0$, *be given by* (A.2.2)*. Then, for any* $t \geq 0$, *we get* $z(t) = \lim_{n\to\infty} h_n(t)$ $\mathbb{P}$ *a.s.*

Let us now show how to use Theorem A.2.1 and Lemma A.2.4 in the setting considered in Chapter 3. Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a continuous time standard Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Let $f \in \mathcal{C}_b(E)$, $G \in \mathcal{C}^+(E)$, and $h \in \mathcal{C}^+(E)$ be such that $0 \leq h(x) \leq G(x)$, $x \in E$. Let us assume ($\mathcal{A}2$) and, for any $T \geq 0$, define

$$Z_T(t) := \exp\left(\int_0^{t\wedge T} f(X_s)ds + 1_{\{t<T\}}G(X_t) + 1_{\{t\geq T\}}h(X_T)\right), \quad t \geq 0. \tag{A.2.4}$$

Recalling Assumption ($\mathcal{A}2$), we get that (A.2.1) is satisfied for the process $(Z_T(t))$, $t \geq 0$, with fixed $T \geq 0$. Next, for any $T \geq 0$, let us define

$$w_T(x) := \inf_{\tau\leq T} \ln \mathbb{E}_x\left[Z_T(\tau)\right], \quad x \in E. \tag{A.2.5}$$

Note that setting $f \equiv g$ and $h \equiv 0$, we get that w_T coincides with $\underline{w}_T$ from (3.3.1). Similarly, for $f \equiv g$ and $h \equiv G$ we get that w_T is equal to $\overline{w}_T$ from (3.3.2).

Now, using Lemma A.2.4, we characterise the Snell envelope of $(Z_T(t))$. The proof is partially based on the argument used in Theorem 1 in Fakeev (1971).

Proposition A.2.5. *For any $T \geq 0$, let $(Z_T(t))$ and w_T be given by* (A.2.4) *and* (A.2.5)*, respectively. Then, for any $t \in [0,T]$ and $x \in E$, we get*

$$\operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}_x \left[Z_T(\tau) | \mathcal{F}_t \right] = e^{\int_0^t f(X_s)ds + w_{T-t}(X_t)} \quad \mathbb{P}_x \text{ a.s.} \tag{A.2.6}$$

Proof. Let us recursively define the family of functions

$$W_1(t,x) := \inf_{\substack{s \in [0,t] \\ s \in \mathbb{Q} \cup \{t\}}} \mathbb{E}_x \left[e^{\int_0^s f(X_u)du + 1_{\{s<t\}} G(X_s) + 1_{\{s \geq t\}} h(X_t)} \right], \quad t \geq 0,\, x \in E,$$

$$W_{n+1}(t,x) := \inf_{\substack{s \in [0,t] \\ s \in \mathbb{Q} \cup \{t\}}} \mathbb{E}_x \left[e^{\int_0^s f(X_u)du} W_n(t-s, X_s) \right], \quad n \in \mathbb{N}_*,\, t \geq 0,\, x \in E,$$

where $\mathbb{Q}$ denotes the set of rational numbers. Also, let us fix some $T \geq 0$ and $x \in E$, and define a version of (h_n) from (A.2.3) corresponding to $(Z_T(t))$ by

$$\begin{aligned} h_0^T(t) &:= Z_T(t), \quad t \in [0,T], \\ h_{n+1}^T(t) &:= \inf_{\substack{s \in [t,T] \\ s \in \mathbb{Q} \cup \{T\}}} \mathbb{E}_x \left[h_n^T(s) | \mathcal{F}_t \right], \quad n \in \mathbb{N},\, t \in [0,T]. \end{aligned}$$

We show that, for any $t \in [0,T]$ and $n \in \mathbb{N}_*$, we get

$$h_n^T(t) = e^{\int_0^t f(X_u)du} W_n(T-t, X_t) \quad \mathbb{P}_x \text{ a.s.} \tag{A.2.7}$$

We proceed by induction. For $n=1$, using the Markov property, we get

$$\begin{aligned} h_1^T(t) &= \inf_{\substack{s \in [t,T] \\ s \in \mathbb{Q} \cup \{T\}}} \mathbb{E}_x \left[e^{\int_0^s f(X_u)du + 1_{\{s<T\}} G(X_s) + 1_{\{s \geq T\}} h(X_T)} | \mathcal{F}_t \right] \\ &= e^{\int_0^t f(X_u)du} \inf_{\substack{s \in [0,T-t] \\ s \in \mathbb{Q} \cup \{T-t\}}} \mathbb{E}_{X_t} \left[e^{\int_0^s f(X_u)du + 1_{\{s<T-t\}} G(X_s) + 1_{\{s \geq T-t\}} h(X_{T-t})} \right] \\ &= e^{\int_0^t f(X_u)du} W_1(T-t, X_t) \quad \mathbb{P}_x \text{ a.s.},\, t \in [0,T]. \end{aligned}$$

Let us now assume that (A.2.7) holds for some $n \in \mathbb{N}_*$. Then, we get

$$\begin{aligned} h_{n+1}^T(t) &= \inf_{\substack{s \in [t,T] \\ s \in \mathbb{Q} \cup \{T\}}} \mathbb{E}_x \left[h_n^T(s) | \mathcal{F}_t \right] \\ &= \inf_{\substack{s \in [t,T] \\ s \in \mathbb{Q} \cup \{T\}}} \mathbb{E}_x \left[e^{\int_0^s f(X_u)du} W_n(T-s, X_s) | \mathcal{F}_t \right] \\ &= e^{\int_0^t f(X_u)du} \inf_{\substack{s \in [0,T-t] \\ s \in \mathbb{Q} \cup \{T-t\}}} \mathbb{E}_{X_t} \left[e^{\int_0^s f(X_u)du} W_n(T-t-s, X_s) \right] \\ &= e^{\int_0^t f(X_u)du} W_{n+1}(T-t, X_t) \quad \mathbb{P}_x \text{ a.s.},\, t \in [0,T], \end{aligned}$$

which concludes the proof of (A.2.7).

Next, noting that $W_{n+1}(t,y) \leq W_n(t,y)$, $n \in \mathbb{N}_*$, $t \in [0,T]$, $y \in E$, we get that the map $W(t,y) := \lim_{n\to\infty} W_n(t,y)$, $t \geq 0$, $y \in E$, is well defined. Thus, using Lemma A.2.4 and (A.2.7), we get

$$\begin{aligned}
\operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}_x\left[Z_T(\tau)|\mathcal{F}_t\right] &= \lim_{n\to\infty} h_n^T(t) \\
&= \lim_{n\to\infty} e^{\int_0^t f(X_u)du} W_n(T-t, X_t) \\
&= e^{\int_0^t f(X_u)du} W(T-t, X_t) \quad \mathbb{P}_x \text{ a.s., } t \in [0,T].
\end{aligned}$$

In particular, setting $z^T(t) := \operatorname{ess\,inf}_{\tau \geq t} \mathbb{E}_x\left[Z_T(\tau)|\mathcal{F}_t\right]$, $t \geq 0$, and using Theorem A.2.1, for any $T \geq 0$ and $x \in E$, we get

$$w_T(x) = \ln \mathbb{E}_x[z^T(0)] = \ln W(T,x).$$

Thus, we get $\operatorname{ess\,inf}_{\tau \geq t} \mathbb{E}_x\left[Z_T(\tau)|\mathcal{F}_t\right] = e^{\int_0^t f(X_u)du + w_{T-t}(X_t)}$, which concludes the proof of (A.2.6). □

Based on Proposition A.2.5, we get the formula for the Snell envelope of the process $(Z_T(t))$. Thus, using Theorem A.2.1, we get an ε-optimal stopping time for the optimal stopping problem associated with w_T. This is summarised in the following corollary.

Corollary A.2.6. *For any $T \geq 0$, let $(Z_T(t))$ and w_T be given by* (A.2.4) *and* (A.2.5)*, respectively. Also, for any $T \geq 0$, let*

$$z_T(t) := e^{\int_0^{t\wedge T} f(X_s)ds + w_{T-t\wedge T}(X_{t\wedge T})}, \quad t \geq 0.$$

Then, for any $\varepsilon > 0$, $T \geq 0$, and $x \in E$, the stopping time

$$\tau_T^\varepsilon := \inf\left\{t \geq 0 : z_T(t) \geq -\varepsilon + Z_T(t)\right\}$$

is ε-optimal for $e^{w_T(x)}$. Also, for any $T \geq 0$ and $x \in E$, the process $(z_T(t))$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale.

Remark A.2.7. Corollary A.2.6 gives an ε-optimal stopping time and a submartingale characterisation of the finite time horizon optimal stopping problem linked to $(Z_T(t))$, $t \geq 0$. Using a similar approach, we might introduce an infinite time horizon version of this process given by

$$Z(t) := \exp\left(\int_0^t g(X_s)ds + G(X_t)\right), \quad t \geq 0,$$

and consider the associated optimal stopping problem. However, in the setting of Chapter 3, we get $g(\cdot) \geq c > 0$ and, consequently, we get that (A.2.1) is not satisfied. Thus, to solve the infinite time horizon problem, we need to use some modified arguments; see Section 3.4 for details. $\diamondsuit$

Let us now characterise an optimal stopping time for the finite time horizon stopping problem associated with the map w_T; see Proposition A.2.8. Note that in the proposition we assume the continuity of $(T,x) \mapsto w_T(x)$. In Theorem 3.3.2, we show that this condition is satisfied for the maps $\underline{w}_T$ and $\overline{w}_T$ given by (3.3.1) and (3.3.2), respectively.

Proposition A.2.8. *For any $T \geq 0$, let w_T be given by (A.2.5). Assume that the map $(T,x) \mapsto w_T(x)$ is jointly continuous. Then, for any $T \geq 0$ and $x \in E$, the stopping time*

$$\tau_T := \inf\{t \geq 0 : w_{T-t}(X_t) = G(X_t)\} \wedge T \tag{A.2.8}$$

is optimal for $w_T(x)$. Moreover, for any $T \geq 0$ and $x \in E$, the process

$$z_T(t) := e^{\int_0^{t\wedge T} f(X_s)ds + w_{T-t\wedge T}(X_{t\wedge T})}, \quad t \geq 0 \tag{A.2.9}$$

is a $\mathbb{P}_x$-submartingale and $(z_T(\tau_T \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale.

Proof. We start with showing the optimality of τ_T. Recall the process $(Z_T(t))$, $T \geq 0$, given by (A.2.4). Using Proposition A.2.5, we get that z_T is the Snell envelope of Z_T. Also, using Corollary A.2.6, we get that, for any $\varepsilon > 0$, $T \geq 0$, and $x \in E$, the process $(z_T(t))$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale and

$$\tau_T^{\varepsilon} := \inf\left\{t \geq 0 : z_T(t) \geq -\varepsilon + Z_T(t)\right\}$$

is an ε-optimal stopping time for $e^{w_T(x)}$. Also, using the fact that

$$z_T(T) = e^{\int_0^T f(X_s)ds + w_0(X_T)} = e^{\int_0^T f(X_s)ds + h(X_T)} = Z_T(T),$$

we get $\tau_T^{\varepsilon} \leq T$. Now, with the help of a suitable limiting procedure, we construct an optimal stopping time. Note that setting

$$\widehat{\tau}_T^{\varepsilon} := \inf\left\{t \geq 0 : e^{w_{T-t}(X_t)} \geq (-\varepsilon)\cdot e^{-\int_0^t f(X_s)ds} + e^{G(X_t)}\right\}, \tag{A.2.10}$$

we get $\tau_T^{\varepsilon} = \widehat{\tau}_T^{\varepsilon} \wedge T$. Thus, noting that $\widehat{\tau}_T^{\varepsilon_1} \geq \widehat{\tau}_T^{\varepsilon_2}$ for $0 \leq \varepsilon_1 \leq \varepsilon_2$, we may define

$$\widehat{\tau}_T := \lim_{\varepsilon\downarrow 0}(\widehat{\tau}_T^{\varepsilon} \wedge T) = \lim_{\varepsilon\downarrow 0} \tau_T^{\varepsilon}.$$

Let us now show that $\widehat{\tau}_T = \tau_T$, where τ_T is given by (A.2.8). For any $\epsilon > 0$, on the event $\{\widehat{\tau}_T^\varepsilon < T\}$, recalling (A.2.10), the continuity of $(T, x) \mapsto w_T(x)$ and $x \mapsto G(x)$, and the right-continuity of (X_t), we get

$$e^{w_{T-\widehat{\tau}_T^\varepsilon}\left(X_{\widehat{\tau}_T^\varepsilon}\right)} \geq (-\varepsilon) \cdot e^{-\int_0^{\widehat{\tau}_T^\varepsilon} f(X_s)ds} + e^{G(X_{\widehat{\tau}_T^\varepsilon})}. \tag{A.2.11}$$

Thus, letting $\varepsilon \downarrow 0$, on the event $\{\widehat{\tau}_T < T\}$, we get $e^{w_{T-\widehat{\tau}_T}(X_{\widehat{\tau}_T})} \geq e^{G(X_{\widehat{\tau}_T})}$. Noting that $w_S(x) \leq G(x)$, for any $x \in E$ and $S \geq 0$, on the event $\{\widehat{\tau}_T < T\}$, we in fact get $w_{T-\widehat{\tau}_T}(X_{\widehat{\tau}_T}) = G(X_{\widehat{\tau}_T})$. Thus, recalling (A.2.8), we get $\tau_T \leq \widehat{\tau}_T$ on the event $\{\widehat{\tau}_T < T\}$. In fact, we get $\tau_T \leq \widehat{\tau}_T$ on Ω, since, on the event $\{\widehat{\tau}_T = T\}$, we get $\tau_T \leq T = \widehat{\tau}_T$. Also, noting that $\tau_T^\varepsilon \leq \tau_T$, $\varepsilon > 0$, and letting $\varepsilon \downarrow 0$, we finally get $\tau_T = \widehat{\tau}_T$.

Now we show that $\tau_T = \widehat{\tau}_T$ is optimal for w_T. Using Fatou's lemma, for any $x \in E$, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}(e^{w_T(x)} + \varepsilon) &\geq \liminf_{\varepsilon\to 0} \mathbb{E}_x \left[e^{\int_0^{\tau_T^\varepsilon} f(X_s)ds + 1_{\{\tau_T^\varepsilon < T\}} G(X_{\tau_T^\varepsilon}) + 1_{\{\tau_T^\varepsilon = T\}} h(X_T)} \right] \\ &\geq \mathbb{E}_x \left[\liminf_{\varepsilon\to 0} e^{\int_0^{\tau_T^\varepsilon} f(X_s)ds + 1_{\{\tau_T^\varepsilon < T\}} G(X_{\tau_T^\varepsilon}) + 1_{\{\tau_T^\varepsilon = T\}} h(X_T)} \right].\end{aligned} \tag{A.2.12}$$

We show that

$$\begin{aligned}&\liminf_{\varepsilon\to 0} e^{\int_0^{\tau_T^\varepsilon} f(X_s)ds + 1_{\{\tau_T^\varepsilon < T\}} G(X_{\tau_T^\varepsilon}) + 1_{\{\tau_T^\varepsilon = T\}} h(X_T)} \\ &\qquad\qquad \geq e^{\int_0^{\widehat{\tau}_T} f(X_s)ds + 1_{\{\widehat{\tau}_T < T\}} G(X_{\widehat{\tau}_T}) + 1_{\{\widehat{\tau}_T = T\}} h(X_T)}.\end{aligned}$$

Note that $\int_0^{\tau_T^\varepsilon} f(X_s)ds \to \int_0^{\widehat{\tau}_T} f(X_s)ds$ as $\varepsilon \downarrow 0$. Also, recalling the monotonicity of $\varepsilon \mapsto \tau_T^\varepsilon$, on the event $A := \bigcup_{n=1}^\infty \{\tau_T^{1/n} = T\}$, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0} \left(1_{\{\tau_T^\varepsilon < T\}} G(X_{\tau_T^\varepsilon}) + 1_{\{\tau_T^\varepsilon = T\}} h(X_T) \right) \\ &\qquad = \lim_{\varepsilon\to 0} 1_{\{\tau_T^\varepsilon = T\}} h(X_T) = 1_{\{\widehat{\tau}_T = T\}} h(X_T) \\ &\qquad\qquad = 1_{\{\widehat{\tau}_T < T\}} G(X_{\widehat{\tau}_T}) + 1_{\{\widehat{\tau}_T = T\}} h(X_T).\end{aligned}$$

Similarly, using the quasi left-continuity of X and recalling that $G \geq h$, on the event $A^c = \bigcap_{n=1}^\infty \{\tau_T^{1/n} < T\}$, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0} \left(1_{\{\tau_T^\varepsilon < T\}} G(X_{\tau_T^\varepsilon}) + 1_{\{\tau_T^\varepsilon = T\}} h(X_T) \right) \\ &\qquad = \lim_{\varepsilon\to 0} G(X_{\tau_T^\varepsilon}) = G(X_{\widehat{\tau}_T}) \\ &\qquad\qquad \geq 1_{\{\widehat{\tau}_T < T\}} G(X_{\widehat{\tau}_T}) + 1_{\{\widehat{\tau} = T\}} h(X_T).\end{aligned}$$

Thus, recalling (A.2.12), for any $x \in E$, we get

$$\lim_{\varepsilon \to 0}(e^{w_T(x)} + \varepsilon) \geq \mathbb{E}_x\left[e^{\int_0^{\widehat{\tau}_T} f(X_s)ds + 1_{\{\widehat{\tau}_T < T\}}G(X_{\widehat{\tau}_T}) + 1_{\{\widehat{\tau}_T = T\}}h(X_T)}\right] \geq e^{w_T(x)}$$

and $\tau_T = \widehat{\tau}_T$ is optimal for w_T.

Finally, let us show the martingale property of $(z_T(\tau_T \wedge t))$, $t \geq 0$. Noting that, for any $t \geq 0$, we get $z_T(\tau_T \wedge t) \leq e^{T\|f\|} \sup_{s\in[0,T]} e^{G(X_s)}$ and using ($\mathcal{A}2$), we get that the process $(z_T(\tau_T \wedge t))$, $t \geq 0$, is uniformly integrable. Thus, recalling that τ_T is optimal for $w_T(x)$, $x \in E$, noting that $1_{\{\tau_T<T\}}G(X_{\tau_T}) = 1_{\{\tau_T<T\}}w_{T-\tau_T}(X_{\tau_T})$ and $h(x) = w_0(x)$, $x \in E$, and using the joint continuity of $(T, x) \mapsto w_T(x)$, we get

$$\begin{aligned}
\mathbb{E}_x[z_T(0)] = e^{w_T(x)} &= \mathbb{E}_x\left[e^{\int_0^{\tau_T} f(X_s)ds + 1_{\{\tau_T<T\}}G(X_{\tau_T}) + 1_{\{\tau_T=T\}}h(X_T)}\right] \\
&= \mathbb{E}_x\left[e^{\int_0^{\tau_T} f(X_s)ds + w_{T-\tau_T}(X_{\tau_T})}\right] \\
&= \mathbb{E}_x\left[\lim_{t\to\infty} e^{\int_0^{\tau_T \wedge t} f(X_s)ds + w_{T-\tau_T\wedge t}(X_{\tau_T\wedge t})}\right] \\
&= \lim_{t\to\infty} \mathbb{E}_x\left[z_T(\tau_T \wedge t)\right], \quad x \in E. \qquad \text{(A.2.13)}
\end{aligned}$$

Also, using the submartingale property of $(z_T(t))$, $t \geq 0$, and Doob's optional stopping theorem, we get that the process $(z_T(\tau_T \wedge t))$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale. In particular, for any $t, h \geq 0$ and $x \in E$, we get

$$\mathbb{E}_x\left[z_T(\tau_T \wedge t)\right] \leq \mathbb{E}_x\left[z_T(\tau_T \wedge (t + h))\right].$$

In fact, recalling (A.2.13), we get $\mathbb{E}_x\left[z_T(\tau_T \wedge t)\right] = \mathbb{E}_x\left[z_T(\tau_T \wedge (t + h))\right]$ for any $t, h \geq 0$. Thus, the process $(z_T(\tau_T \wedge t))$, $t \geq 0$, is a submartingale with a constant expectation, hence it is a martingale, which concludes the proof. $\square$

To verify the joint continuity condition from Proposition A.2.8, we use the following simple calculus lemma; this is extensively used in the proof of Theorem 3.3.2.

Lemma A.2.9. *Let us consider the map $h\colon \mathbb{R}_+ \times E \to \mathbb{R}$. Suppose that, for any $t_0 \in \mathbb{R}_+$ and $x_0 \in E$, the maps $t \mapsto h(t, x_0)$ and $x \mapsto h(t_0, x)$ are continuous. Also, suppose that, for any $x_0 \in E$, the map $t \mapsto h(t, x_0)$ is monotonic (increasing or decreasing). Then, the map $(t, x) \mapsto h(t, x)$ is jointly continuous.*

Proof. For brevity, we assume that, for any $x_0 \in E$, the map $t \mapsto h(t, x_0)$ is increasing; the remaining case could be treated using similar logic. Let $t \geq 0$

and $x \in E$. Also, let the sequences $(t_n) \subset \mathbb{R}_+$ and $(x_n) \subset E$ be such that $t_n \to t$ and $x_n \to x$ as $n \to \infty$. Since $(t_n) \subset \mathbb{R}_+$, we may choose a subsequence (for brevity still denoted by (t_n)) such that the convergence of (t_n) is monotonic. Then, using the continuity of $x \mapsto h(t_0, x)$ and $t \mapsto h(t, x_0)$, and the monotonicity of $t \mapsto h(t, x_0)$, we get that the map $H_n \colon E \ni x \mapsto h(t_n, x)$, $n \in \mathbb{N}$, is continuous and converges monotonically as $n \to \infty$ to the continuous map $H \colon E \ni x \mapsto h(t, x)$. Thus, from Dini's theorem, we get that the convergence of H_n to H is uniform in x from compact sets; see e.g. Theorem 7.13 in Rudin (1976) for details. Thus, we get

$$\begin{aligned} |h(t_n, x_n) - h(t, x)| &\leq |h(t_n, x_n) - h(t, x_n)| + |h(t, x_n) - h(t, x)| \\ &\leq |H_n(x_n) - H(x_n)| + |H(x_n) - H(x)| \to 0, \quad n \to \infty, \end{aligned}$$

which concludes the proof. □

A.3 Dyadic impulse control

In this section, we present some results related to the long-run dyadic impulse control problems. In particular, we show the existence of a solution to a suitable form of the impulse control Bellman equation and its link with the optimal value of the problem. This section could be seen as an excerpt from Pitera and Stettner (2021).

Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a continuous time standard $\mathcal{C}_b$-Feller-Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Next, for any $m \in \mathbb{N}$, we set $\delta_m := \frac{1}{2^m}$. In this section we consider impulse control strategies $V \in \mathbb{V}^m$ with impulse times restricted to the dyadic time-grid $\{k\delta_m \colon k \in \mathbb{N}\} \cup \{+\infty\}$. As in Section 2.2, with a starting point $x \in E$ and an impulse control strategy $V = (\tau_i, \xi_i)_{i=1}^{\infty} \in \mathbb{V}^m$, we associate the controlled process, the corresponding probability measure, and the expectation operator denoted by Y, $\mathbb{P}_{(x,V)}$, and $\mathbb{E}_{(x,V)}$, respectively. Throughout this section we assume that after-impulse states are restricted to some fixed compact set $U \subseteq E$, i.e. $\xi_i \in U$, $i \in \mathbb{N}_*$. Also, we recall the entropic utility measure J^γ, $\gamma < 0$, defined by (2.3.1). Note that, for any $\gamma < 0$, $x \in E$, and $V \in \mathbb{V}$, by J^γ_x and $J^\gamma_{(x,V)}$ we denote the corresponding versions of J^γ, where, in (2.3.1), the expectation operator $\mathbb{E}$ is replaced by $\mathbb{E}_x$ and $\mathbb{E}_{(x,V)}$, respectively.

Let us now state some fundamental properties of the long-run dyadic impulse control problem; see Proposition A.3.1. In particular, we get the existence of a solution to the Bellman equation and the optimal value of the problem; see (A.3.3) and (A.3.4). For the proof of Proposition A.3.1, we refer to Proposition 3.4 and Proposition 4.3 in Pitera and Stettner (2021).

Proposition A.3.1. *Let $\gamma < 0$ and $m \in \mathbb{N}$. Assume the following conditions:*

(1) *We have $\widehat{f} \in \mathcal{C}_b(E)$.*

(2) *We have $\widehat{c} \in \mathcal{C}(E \times U)$. Also, there exists $\widehat{c}_0 < 0$ such that, for any $x \in E$ and $\xi \in U$, we have*

$$\widehat{c}(x,\xi) \leq \widehat{c}_0. \tag{A.3.1}$$

Moreover, we have

$$\sup_{x\in E}\inf_{\xi\in U} |\widehat{c}(x,\xi)| < \infty. \tag{A.3.2}$$

(3) *There exists $a_m > 0$ and a probability measure ν_m on $(E,\mathcal{E})$ such that $\nu_m(U) > 0$ and*

$$\inf_{x\in U} \mathbb{P}_x[X_{\delta_m} \in A] \geq a_m \nu_m(A), \quad A \in \mathcal{E}.$$

Then, there exists a unique (up to an additive constant) function $\widehat{u}_m^\gamma \in \mathcal{C}_b(E)$ and a constant $\lambda_m^\gamma \in \mathbb{R}$ satisfying

$$\widehat{u}_m^\gamma(x) + \lambda_m^\gamma = \max\Bigg(\sup_{\xi\in U}\left(J_\xi^\gamma\left(\int_0^{\delta_m}\widehat{f}(X_s)ds + \widehat{u}_m^\gamma(X_{\delta_m})\right) + \widehat{c}(x,\xi)\right), \\ J_x^\gamma\left(\int_0^{\delta_m}\widehat{f}(X_s)ds + \widehat{u}_m^\gamma(X_{\delta_m})\right)\Bigg). \tag{A.3.3}$$

Also, for any $x \in E$, we get

$$\lambda_m^\gamma/\delta_m = \sup_{V\in\mathbb{V}^m}\liminf_{T\to\infty}\frac{1}{T}J^\gamma_{(x,V)}\left(\int_0^T \widehat{f}(Y_s)ds + \sum_{i=1}^\infty 1_{\{\tau_i\leq T\}}\widehat{c}(Y_{\tau_i^-},\xi_i)\right). \tag{A.3.4}$$

Remark A.3.2. It should be noted that in Pitera and Stettner (2021) a more general setting is considered. In particular, in that paper, the maps $\widehat{f}$ and $\widehat{c}$ could be unbounded provided that they satisfy some growth condition expressed in terms of a suitable weight norm. In Proposition A.3.1, to simplify the narrative, we use a bounded setting that corresponds to the weight function identically equal to 0. In particular, we get that Assumption (A.3) from Pitera and Stettner (2021) is trivially satisfied. ♢

Let us now show how to use Proposition A.3.1 in the setting introduced in Section 4.1. In the following, $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a continuous time standard $\mathcal{C}_b$-Feller-Markov process on $(\Omega,\mathcal{F},\mathbb{F})$ with values in $(E,\mathcal{E})$, and we assume ($\mathcal{B}$1)–($\mathcal{B}$5).

Proposition A.3.3. *Let the map J be given by (4.3.2). Then, for any $m \in \mathbb{N}$, there exists a unique (up to an additive constant) function $u_m \in \mathcal{C}_b(E)$ and a constant $\lambda_m \in \mathbb{R}$ satisfying*

$$e^{u_m(x)} = \min \left(\inf_{\xi \in U} e^{c(x,\xi)} \, \mathbb{E}_\xi \left[e^{\int_0^{\delta_m} (f(X_s)-\lambda_m)ds + u_m(X_{\delta_m})} \right], \right. \\ \left. \mathbb{E}_x \left[e^{\int_0^{\delta_m} (f(X_s)-\lambda_m)ds + u_m(X_{\delta_m})} \right] \right). \quad \text{(A.3.5)}$$

Also, we get

$$\lambda_m = \inf_{V \in \mathbb{V}^m} J(x, V), \quad x \in E. \quad \text{(A.3.6)}$$

Proof. We transform the associated impulse control problem into the setting considered in Proposition A.3.1. Let us fix some $m \in \mathbb{N}$ and $\gamma < 0$.Next, recall the maps $f \in \mathcal{C}_b^+(E)$ and $c \in \mathcal{C}_b^+(E \times U)$ satisfying ($\mathcal{B}2$)–($\mathcal{B}3$). Also, let us define $\widehat{f}(\cdot) := \frac{f(\cdot)}{\gamma}$ and $\widehat{c}(\cdot,\text{-}) := \frac{c(\cdot,\text{-})}{\gamma}$. We show that the assumptions of Proposition A.3.1 are satisfied for m, γ, $\widehat{f}$, $\widehat{c}$, and the Markov process X. First, note that directly from the boundedness of f, we get that (1) is satisfied. Second, note that from the continuity of c, we get $\widehat{c} \in \mathcal{C}(E \times U)$. Moreover, recalling $c_0 > 0$ from Assumption ($\mathcal{B}3$) and setting $\widehat{c}_0 := \frac{c_0}{\gamma}$, we get (A.3.1). Also, from the boundedness of c, we get (A.3.2). Thus, we get that (2) is satisfied. Finally, recalling that ($\mathcal{B}5$) implies ($\mathcal{B}5$a), we get that (3) is also satisfied.

Then, using Proposition A.3.1, we get that there exists a unique (up to an additive constant) function $\widehat{u}_m^\gamma \in \mathcal{C}_b(E)$ and a constant $\lambda_m^\gamma \in \mathbb{R}$ satisfying (A.3.3). In particular, recalling (2.3.1), we get

$$\widehat{u}_m^\gamma(x) + \lambda_m^\gamma = \max \left(\sup_{\xi \in U} \left(\frac{1}{\gamma} \ln \mathbb{E}_\xi \left[\int_0^{\delta_m} \gamma \widehat{f}(X_s) ds + \gamma \widehat{u}_m^\gamma(X_{\delta_m}) \right] + \widehat{c}(x,\xi) \right), \right. \\ \left. \frac{1}{\gamma} \ln \mathbb{E}_x \left[\int_0^{\delta_m} \gamma \widehat{f}(X_s) ds + \gamma \widehat{u}_m^\gamma(X_{\delta_m}) \right] \right). \quad \text{(A.3.7)}$$

Define $u_m(\cdot) := \gamma \widehat{u}_m^\gamma(\cdot)$ and note that $u_m \in \mathcal{C}_b(E)$. Also, let $\lambda_m := \frac{\gamma \lambda_m^\gamma}{\delta_m}$. Thus, multiplying both sides of (A.3.7) by $\gamma < 0$ and taking the exponential function, we get that the pair (u_m, λ_m) satisfies (A.3.5). In fact, this solution is unique (up to an additive constant for u_m) since the solution to (A.3.3) is unique (up to an additive constant for $\widehat{u}_m^\gamma$) and we get a one-to-one correspondence between (A.3.3) and (A.3.5). Also, from (A.3.4), for any $x \in E$, we get

$$\lambda_m^\gamma / \delta_m = \sup_{V \in \mathbb{V}^m} \liminf_{T \to \infty} \frac{1}{T} \frac{1}{\gamma} \ln E_{(x,V)} \left[\int_0^T \gamma \widehat{f}(Y_s) ds + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} \gamma \widehat{c}(Y_{\tau_i^-}, \xi_i) \right].$$

Thus, multiplying both sides by $\gamma < 0$, we get (A.3.6), which concludes the proof. $\square$

A.4 Multiplicative Poisson Equation

In this section, we discuss the properties of a solution to the Multiplicative Poisson Equation (MPE). Also, we introduce a change of measure technique based on a solution to MPE and show that this transformation preserves some important properties of Feller–Markov processes. This is extensively used in Section 4.3.

This section is based mainly on Stettner (1989), where a useful characterisation of the existence of a solution to MPE is established. Also, we refer to Fleming et al. (1987); Balaji and Meyn (2000), and Kontoyiannis and Meyn (2003) for the connection with Donsker–Varadhan theory and large deviations. The existence of a solution to MPE using a span contraction approach was studied in Di Masi and Stettner (1999) and with the help of a chain splitting technique in Di Masi and Stettner (2007). The complete characterisation of the existence for the finite state space Markov chains can be found in Cavazos-Cadena and Hernández-Hernández (2009). The connection with the principal eigenvalue of a suitable diffusion operator is discussed in Arapostathis and Biswas (2018).

A.4.1 Existence of a solution to MPE

Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a continuous-time standard $\mathcal{C}_b$-Feller-Markov process on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Also, let $f \in \mathcal{C}_b(E)$ and recall the semigroup $(\mathcal{P}_t^f)_{t\geq 0}$ defined by (2.1.5). With this semigroup we associate its type $r(f)$ given by

$$r(f) := \inf_{t>0} \frac{1}{t} \ln \sup_{x\in E} \mathcal{P}_t^f \mathbb{1}(x), \tag{A.4.1}$$

where $\mathbb{1}$ denotes the function identically equal to 1. Directly from the definition, we get $-\|f\| \leq r(f) \leq \|f\|$. Also, using Fekete's subadditive lemma, we may show that

$$r(f) = \lim_{t\to\infty} \frac{1}{t} \ln \sup_{x\in E} \mathcal{P}_t^f \mathbb{1}(x), \tag{A.4.2}$$

see Chapter 1.4.2 in Kato (1995) for details and Proposition 1 in Stettner (1989) for alternative characterisations of $r(f)$. Based on (A.4.2), we get that $r(f)$ quantifies the long-term behaviour of the exponential semigroup $(\mathcal{P}_t^f)$, i.e. we get that $\sup_{x\in E} \mathcal{P}_t^f \mathbb{1}(x)$ is asymptotically equivalent to $e^{r(f)t}$ as $t \to \infty$.

This section focuses on the properties of a solution to the Multiplicative Poisson Equation, i.e. a map $v \in \mathcal{C}_b(E)$ satisfying

$$v(x) = \ln \mathbb{E}_x \left[\exp \left(\int_0^t (f(X_s) - r(f))ds + v(X_t) \right) \right], \quad x \in E,\, t \geq 0. \quad \text{(A.4.3)}$$

This equation could be seen as an uncontrolled version of the impulse control Bellman equation (4.3.3). As we show in this section, with a solution to (A.4.3) we may associate a change of measure transformation that simplifies some optimisation problems.

Let us now provide exemplary sufficient conditions for the existence of a solution to (A.4.3). We refer to Lemma 3.2 in Sadowy and Stettner (2002) for the proof; see also Proposition 6 in Stettner (1989) and Proposition 2.7 in Di Masi and Stettner (1999) for some other conditions.

Theorem A.4.1. *Suppose that one of the following conditions holds:*

(1) *There exists $t_0 > 0$ such that, for any $t \in (0, t_0)$, we may find a probability measure ν_t on $(E, \mathcal{E})$, a density $p_t : E \times E \to \mathbb{R}_+$, and constants $0 < a_t \leq b_t < \infty$, such that*

$$\mathbb{P}_x[X_t \in A] = \int_A p_t(x,y)\nu_t(dy), \quad A \in \mathcal{E} \quad \text{(A.4.4)}$$

and $a_t \leq p_t(x,y) \leq b_t$, $x, y \in E$, or:

(2) *There exists $t_0 > 0$ such that, for any $t \in (0, t_0)$, we get*

$$\kappa_t := \sup_{x,y \in E} \sup_{A \in \mathcal{E}} (\mathbb{P}_x[X_t \in A] - \mathbb{P}_y[X_t \in A]) < 1$$

and $\kappa_t e^{t\|f\|_{span}} < 1$, where $\|f\|_{span} := \sup_{x,y \in E}(f(x) - f(y))$.

Then, there exists a unique up to an additive constant function $v \in \mathcal{C}_b(E)$ satisfying (A.4.3).

Remark A.4.2. Note that in (A.4.3) we require the formula to be satisfied for any $t \geq 0$. It can be shown that, in fact, we only need a seemingly weaker condition. Namely, suppose that there exists $v \in \mathcal{C}_b(E)$ such that (A.4.3) is satisfied for any $x \in E$ and $t \in [0, t_0)$ with some $t_0 > 0$. Then, for any $t \geq t_0$, we may find $n \in \mathbb{N}_*$ such that $t/n < t_0$. Hence, iterating (A.4.3) and using the Markov property of X, we get

$$\begin{aligned} v(x) &= \ln \mathbb{E}_x \left[e^{\int_0^{t/n} (f(X_s) - r(f))ds + v(X_{t/n})} \right] \\ &= \ln \mathbb{E}_x \left[e^{\int_0^{2t/n} (f(X_s) - r(f))ds + v(X_{2t/n})} \right] \\ &= \ln \mathbb{E}_x \left[e^{\int_0^{t} (f(X_s) - r(f))ds + v(X_{t})} \right], \quad x \in E. \end{aligned}$$

Thus, we get that (A.4.3) is satisfied for any $t \geq 0$. $\Diamond$

Remark A.4.3. In our framework, we require a solution to (A.4.3) to be continuous and bounded, which facilitates the analysis in Chapter 4. In a general setting, one may look for $v \in \mathcal{M}(E)$ and $\mu \in \mathbb{R}$ satisfying

$$v(x) = \ln \mathbb{E}_x \left[\exp \left(\int_0^t (f(X_s) - \mu) ds + v(X_t) \right) \right], \quad x \in E,\, t \geq 0; \quad \text{(A.4.5)}$$

provided that the expectation is well defined. It can be shown that if a pair (v, μ) satisfies (A.4.5) and v is bounded, then $\mu = r(f)$, i.e. the constant μ is uniquely defined; see the discussion following Corollary 2 in Stettner (1989). An unbounded solution to (A.4.5) can be found e.g. in the example following Remark 2 in Di Masi and Stettner (2007). $\Diamond$

A.4.2 Change of measure transformation

The existence of a solution to (A.4.3) facilitates the use of the change of measure technique that simplifies certain stochastic control problems. More specifically, let $v \in \mathcal{C}_b(E)$ be a solution to (A.4.3) and, for any $x \in E$, let us define the process

$$Y_t(x) := e^{-v(x)} e^{\int_0^t (f(X_s) - r(f))ds + v(X_t)}, \quad t \geq 0,\, x \in E. \quad \text{(A.4.6)}$$

In Proposition A.4.4 we show that $(Y_t(x))$, $t \geq 0$, is a martingale with unit expectation. Thus, for any $x \in E$, we may define a probability measure $\mathbb{Q}_x$ given via a Radon–Nikodym derivative

$$d\mathbb{Q}_x\big|_{\mathcal{F}_t} := Y_t(x) d\mathbb{P}_x\big|_{\mathcal{F}_t}, \qquad t \geq 0. \quad \text{(A.4.7)}$$

In fact, (A.4.7) defines a consistent family of measures on $(\Omega, \mathcal{F}_t)$, $t \geq 0$, which, under some technical assumptions (e.g. probability space being a canonical space of càdlàg processes) can be extended to $(\Omega, \mathcal{F})$; see e.g. Section 3.5 in Karatzas and Shreve (1998*a*) and Theorem 4.2 in Chapter V of Parthasarathy (1967) for details. In the following we assume that this extension is possible and satisfies (A.4.7). Also, by $\mathbb{E}_x^{\mathbb{Q}}$, $x \in E$, we denote the expectation operator corresponding to $\mathbb{Q}_x$.

In Proposition A.4.4 we show some basic properties of the measures $\mathbb{Q}_x$, $x \in E$. The proof is relatively standard yet hardly accessible in the literature, and we include it for completeness.

Proposition A.4.4. *Let $v \in \mathcal{C}_b(E)$ be a solution to (A.4.3). Also, for any $x \in E$, let $(Y_t(x))$, $t \geq 0$, and $\mathbb{Q}_x$ be given by (A.4.6) and (A.4.7), respectively. Then:*

(1) *For any $x \in E$, the process $(Y_t(x))$, $t \geq 0$, is a $\mathbb{P}_x$-martingale and* $\mathbb{E}_x\left[Y_t(x)\right] = 1$.

(2) *For any $x \in E$, $\lambda \in \mathbb{R}$, $G \in \mathcal{C}_b(E)$, and $\tau \in \mathcal{T}_{x,b}$, we get*

$$\mathbb{E}_x\left[e^{\int_0^\tau (f(X_s)-\lambda)ds+G(X_\tau)}\right] = e^{v(x)}\,\mathbb{E}_x^{\mathbb{Q}}\left[e^{(r(f)-\lambda)\tau+G(X_\tau)-v(X_\tau)}\right]. \quad \text{(A.4.8)}$$

(3) *For any $h \in \mathcal{C}_b(E)$, we get*

$$\mathbb{E}_x^{\mathbb{Q}}\left[h(X_{t+u})|\mathcal{F}_t\right] = \mathbb{E}_{X_t}^{\mathbb{Q}}\left[h(X_u)\right], \quad x \in E,\, t, u \geq 0.$$

(4) *If a process $(z(t))$, $t \geq 0$, is a $\mathbb{Q}_x$-submartingale (respectively, martingale, supermartingale), then $\widetilde{z}(t) := e^{\int_0^t (f(X_s)-r(f))ds+v(X_t)}z(t)$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale (respectively, martingale, supermartingale). Also, if a process $(z(t))$, $t \geq 0$, is a $\mathbb{P}_x$-submartingale (respectively, martingale, supermartingale), then $\widetilde{z}(t) := e^{-\int_0^t (f(X_s)-r(f))ds-v(X_t)}z(t)$, $t \geq 0$, is a $\mathbb{Q}_x$-submartingale (respectively, martingale, supermartingale).*

Proof. For transparency, we prove the claims point by point.

Proof of (1). Using the Markov property, for any $t, u \geq 0$, we get

$$\begin{aligned}\mathbb{E}_x\left[Y_{t+u}(x) \mid \mathcal{F}_t\right] &= e^{\int_0^t (f(X_s)-r(f))ds}\mathbb{E}_x\left[e^{\int_t^{t+u}(f(X_s)-r(f))ds+v(X_{t+u})}\middle|\mathcal{F}_t\right]\\ &= e^{\int_0^t (f(X_s)-r(f))ds}\mathbb{E}_{X_t}\left[e^{\int_0^u (f(X_s)-r(f))ds+v(X_u)}\right]\\ &= e^{\int_0^t (f(X_s)-r(f))ds}e^{v(X_t)} = Y_t(x), \quad x \in E.\end{aligned}$$

Also, directly from (A.4.3) and (A.4.6), we get $\mathbb{E}_x\left[Y_t(x)\right] = 1$, $x \in E$, $t \geq 0$, which concludes the proof of this point.

Proof of (2). By Doob's optional stopping theorem, the boundedness of τ, and point (1), we get $\mathbb{E}_x\left[Y_\tau(x)\right] = 1$. Thus, we get

$$\begin{aligned}&\mathbb{E}_x\left[e^{\int_0^\tau (f(X_s)-\lambda)ds+G(X_\tau)}\right]\\ &\quad= \mathbb{E}_x\left[Y_\tau(x)e^{v(x)}e^{-\int_0^\tau (f(X_s)-r(f))ds-v(X_\tau)}e^{\int_0^\tau (f(X_s)-\lambda)ds+G(X_\tau)}\right]\\ &\quad= e^{v(x)}\,\mathbb{E}_x^{\mathbb{Q}}\left[e^{(r(f)-\lambda)\tau+G(X_\tau)-v(X_\tau)}\right],\end{aligned}$$

which concludes the proof of this point.

Proof of (3). Let us fix $h \in \mathcal{C}_b(E)$, $x \in E$, and $t, u \geq 0$. Using the Bayes rule for conditional expectation (see e.g. Lemma 5.3 in Chapter 3 of Karatzas and Shreve (1998*a*)), we get

$$\begin{aligned}
\mathbb{E}_x^{\mathbb{Q}}\left[h(X_{t+u})|\mathcal{F}_t\right] &= \frac{1}{\mathbb{E}_x\left[Y_{t+u}(x)|\mathcal{F}_t\right]}\mathbb{E}_x\left[Y_{t+u}(x)h(X_{t+u})|\mathcal{F}_t\right] \\
&= \frac{Y_t(x)e^{-v(X_t)}\mathbb{E}_x\left[e^{\int_t^{t+u}(f(X_s)-r(f))ds+v(X_{t+u})}h(X_{t+u})|\mathcal{F}_t\right]}{Y_t(x)} \\
&= e^{-v(X_t)}\mathbb{E}_{X_t}\left[e^{\int_0^u(f(X_s)-\mu)ds+v(X_u)}h(X_u)\right] \\
&= \mathbb{E}_{X_t}^{\mathbb{Q}}\left[h(X_u)\right],
\end{aligned}$$

which concludes the proof of this point.

Proof of (4). We show the proof only for $(z(t))$ being a $\mathbb{Q}_x$-submartingale; the proofs for the remaining claims are similar and omitted for brevity. Let $t, h \geq 0$ and $x \in E$. As in point (3), using the Bayes rule for conditional expectation, we get

$$\frac{1}{\mathbb{E}_x\left[Y_{t+h}(x)|\mathcal{F}_t\right]}\mathbb{E}_x\left[Y_{t+h}(x)z(t+h)|\mathcal{F}_t\right] = \mathbb{E}_x^{\mathbb{Q}}\left[z(t+h)|\mathcal{F}_t\right] \geq z(t).$$

Hence, recalling the martingale property of $(Y_t(x))$, we get

$$\mathbb{E}_x\left[Y_{t+h}(x)z(t+h)|\mathcal{F}_t\right] \geq z(t)Y_t(x).$$

Consequently, we get

$$\begin{aligned}
e^{-v(x)}\mathbb{E}_x\left[\widetilde{z}(t+h)|\mathcal{F}_t\right] &= \mathbb{E}_x\left[e^{-v(x)}e^{\int_0^{t+h}(f(X_s)-r(f))ds+v(X_{t+h})}z(t+h)|\mathcal{F}_t\right] \\
&= \mathbb{E}_x\left[Y_{t+h}(x)z(t+h)|\mathcal{F}_t\right] \\
&\geq z(t)Y_t(x) = e^{-v(x)}\widetilde{z}(t),
\end{aligned}$$

which concludes the proof. □

Let us now show that the change of measure preserves the Feller property. To simplify the notation, we define the semigroup $(\mathcal{P}_t^{\mathbb{Q}})_{t\geq 0}$ by

$$\mathcal{P}_t^{\mathbb{Q}}h(x) := \mathbb{E}_x^{\mathbb{Q}}[h(X_t)], \quad t \geq 0,\ h \in \mathcal{M}_b(E),\ x \in E; \tag{A.4.9}$$

note that $\mathcal{P}_t^{\mathbb{Q}}$ could be seen as a version of $\mathcal{P}_t$ from (2.1.4) corresponding to the family $(\mathbb{Q}_x)$. Also, for any $t, r > 0$ and a compact set $\Gamma \subset E$, we define

$$M_\Gamma(t,r) := \sup_{x\in\Gamma}\mathbb{P}_x[\sup_{s\in[0,t]}\rho(X_s,x) \geq r], \tag{A.4.10}$$

$$M_\Gamma^{\mathbb{Q}}(t,r) := \sup_{x\in\Gamma}\mathbb{Q}_x[\sup_{s\in[0,t]}\rho(X_s,x) \geq r]. \tag{A.4.11}$$

Proposition A.4.5. *Let $v \in \mathcal{C}_b(E)$ be a solution to (A.4.3) and let $(\mathbb{Q}_x)_{x\in E}$ be given by (A.4.7). Next, let the semigroups $(\mathcal{P}_t)_{t\geq 0}$ and $(\mathcal{P}_t^{\mathbb{Q}})_{t\geq 0}$ be given by (2.1.4) and (A.4.9), respectively. Also, let $t_0, r_0 > 0$, and $\Gamma \subset E$ be a compact set. Then:*

(1) *If the semigroup $(\mathcal{P}_t)_{t\geq 0}$ is $\mathcal{C}_b$-Feller (respectively, $\mathcal{C}_0$-Feller), then the semigroup $(\mathcal{P}_t^{\mathbb{Q}})_{t\geq 0}$ is also $\mathcal{C}_b$-Feller (respectively, $\mathcal{C}_0$-Feller).*

(2) *If we get $\lim_{t\to 0} M_\Gamma(t, r_0) = 0$, then we get $\lim_{t\to 0} M_\Gamma^{\mathbb{Q}}(t, r_0) = 0$. Also, if we get $\lim_{r\to\infty} M_\Gamma(t_0, r) = 0$, then we get $\lim_{r\to\infty} M_\Gamma^{\mathbb{Q}}(t_0, r) = 0$.*

Proof. To prove (1), note that, for any $h \in \mathcal{C}_b(E)$, we get

$$\mathcal{P}_t^{\mathbb{Q}} h(x) = e^{-v(x)}\mathbb{E}_x\left[e^{\int_0^t (f(X_s)-r(f))ds + v(X_t)} h(X_t)\right], \quad t \geq 0,\, x \in E.$$

Thus, recalling the continuity of v and using Proposition 2.1.8, we get that the property $\mathcal{P}_t\mathcal{C}_b(E) \subset \mathcal{C}_b(E)$, $t > 0$, implies $\mathcal{P}_t^{\mathbb{Q}}\mathcal{C}_b(E) \subset \mathcal{C}_b(E)$, $t > 0$. Also, from the boundedness of v and f, we get

$$|\mathcal{P}_t^{\mathbb{Q}} h(x)| \leq e^{2\|v\| + t\|f - r(f)\|} |\mathcal{P}_t h(x)|, \quad t \geq 0,\, h \in \mathcal{C}_b(E),\, x \in E.$$

Thus, from $\mathcal{P}_t\mathcal{C}_0(E) \subset \mathcal{C}_0(E)$, $t > 0$, we get $\mathcal{P}_t^{\mathbb{Q}}\mathcal{C}_0(E) \subset \mathcal{C}_0(E)$, $t > 0$, which concludes the proof of (1).

For the proof of (2), it is enough to note that, for any $t, r > 0$, we get

$$\begin{aligned} M_\Gamma^{\mathbb{Q}}(t, r) &= \sup_{x\in\Gamma} e^{-v(x)}\mathbb{E}_x\left[e^{\int_0^t (f(X_s)-r(f))ds + v(X_t)} 1_{\{\sup_{s\in[0,t]}\rho(X_s, x)\geq r\}}\right] \\ &\leq e^{2\|v\| + t\|f - r(f)\|} M_\Gamma(t, r). \end{aligned}$$

Thus, letting $t \to 0$ or $r \to \infty$, we conclude the proof. $\square$

In Theorem A.4.6 we show that the process $(X_t)_{t\geq 0}$ is standard Markov under the family of measures $(\mathbb{Q}_x)_{x\in E}$. Also, we get that the change of measure preserves the conditions from Assumptions ($\mathcal{A}4$) and ($\mathcal{B}4$). This is extensively used in Section 4.3.

Theorem A.4.6. *Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a standard $\mathcal{C}_b$-Feller–Markov on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Assume that, for any $t_0 > 0$, $r_0 > 0$, and a compact set $\Gamma \subset E$, we get*

$$\lim_{t\to 0} M_\Gamma(t, r_0) = 0 \quad \text{and} \quad \lim_{r\to\infty} M_\Gamma(t_0, r) = 0,$$

where M_Γ is given by (A.4.10). Then, $((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ is a standard $\mathcal{C}_b$-Feller–Markov on $(\Omega, \mathcal{F}, \mathbb{F})$ with values in $(E, \mathcal{E})$. Also, for any $t_0 > 0$, $r_0 > 0$, and a compact set $\Gamma \subset E$, we get

$$\lim_{t\to 0} M_\Gamma^{\mathbb{Q}}(t, r_0) = 0 \quad \textit{and} \quad \lim_{r\to\infty} M_\Gamma^{\mathbb{Q}}(t_0, r) = 0, \tag{A.4.12}$$

where M_Γ^Q is given by (A.4.11).

Proof. Using Proposition A.4.4, we get that $((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ satisfies the Markov property for deterministic stopping times. Moreover, using Proposition A.4.5, we get that X is $\mathcal{C}_b$-Feller with respect to $(\mathbb{Q}_x)$. Thus, recalling the right-continuity of X and using Theorem 5.10 in Dynkin (1961), we get that $((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ is strong Markov. Also, using Proposition A.4.4, we get that (A.4.12) is satisfied. In particular, we get that the condition $M(\Gamma)$ from Dynkin (1961) is satisfied for any compact set $\Gamma \subset E$. Thus, using Theorem 6.7 in Dynkin (1961), we get that $((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ is quasi left-continuous. Hence, $((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ is a standard Markov process, which concludes the proof. $\square$

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Index of notation

Basic symbols and abbreviations

A^c	Complement of a set A
$\mathbb{N}$	Non-negative integers $\{0, 1, 2, 3, \ldots\}$
$\mathbb{N}_*$	Positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{R}_+$	Non-negative real numbers $[0, \infty)$
$x \wedge y$	Minimum of two real numbers $\min(x, y)$
a.s.	Almost surely
$\inf \emptyset := +\infty$	Infimum of the empty set
$\sum_{i=0}^{-1}(\cdot) := 0$	Value of an "empty" sum
$\blacklozenge$	End of example
$\lozenge$	End of remark

Notation

$\mathcal{C}_b^+(A)$	10	$\mathcal{C}_b(A) \cap \mathcal{C}^+(A)$
$\mathcal{C}_0(A)$	10	Family of continuous real-valued functions on a metric space A vanishing at infinity
$\\|f\\|$	10	Supremum norm for $f \in \mathcal{M}_b(A)$
$(E, \mathcal{E})$	10	State space with the σ-field
ρ, $\\|\cdot\\|$	10	Metric and norm-like function on E
$\mathbb{P}_x$, $\mathbb{E}_x$	12	Conditional distribution and the corresponding expectation operator for a Markov process starting from $x \in E$
$\mathcal{T}$	12	Family of stopping times
$\mathcal{T}_x$	12	Family of $\mathbb{P}_x$ a.s. finite stopping times
$\mathcal{T}_{x,b}$	12	Family of $\mathbb{P}_x$ a.s. bounded stopping times
$\mathcal{T}^m$	12	Family of stopping times with values in $\{k\delta_m \colon k \in \mathbb{N}\} \cup \{+\infty\}$, where $\delta_m := \frac{1}{2^m}$
$\mathcal{T}_x^m$	12	$\mathcal{T}^m \cap \mathcal{T}_x$
$\mathcal{T}_{x,b}^m$	12	$\mathcal{T}^m \cap \mathcal{T}_{x,b}$
$\mathcal{P}_t$	13	Transition semigroup corresponding to a Markov process X
$\mathcal{P}_t^f$	15	Exponential semigroup corresponding to a Markov process X and some function $f \in \mathcal{C}_b(E)$
$\mathbb{V}$	17	Family of impulse control strategies
$\mathbb{V}^m$	17	Family of impulse control strategies with impulse times on the dyadic time-grid $\{k\delta_m \colon k \in \mathbb{N}\} \cup \{+\infty\}$, where $\delta_m := \frac{1}{2^m}$
$\mathbb{V}_n$	17	Family of impulse control strategies with at most n impulses
$\mathbb{V}_n^m$	17	$\mathbb{V}^m \cap \mathbb{V}_n$
$\mathbb{P}_{(x,V)}, \mathbb{E}_{(x,V)}$	17	Measure and the corresponding expectation operator for the controlled process starting at $x \in E$ and a strategy $V \in \mathbb{V}$
$\mathbb{Q}_x$, $\mathbb{E}_x^{\mathbb{Q}}$	152	Measure and the corresponding expectation operator induced by a solution to the Multiplicative Poisson Equation